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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 63 (2017) 197–202 www.elsevier.com/locate/endm

On the power graph of the direct product of two groups

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Abstract

The power graph $P(G)$ of a finite group G is the graph with vertex set G and two distinct vertices are adjacent if either of them is a power of the other. Here we show that the power graph $P(G_1 \times G_2)$ of the direct product of two groups G_1 and G_2 is not isomorphic to either of the direct, cartesian and normal product of their power graphs $P(G_1)$ and $P(G_2)$. A new product of graphs, namely generalized product, has been introduced and we prove that the power graph $P(G_1 \times G_2)$ is isomorphic to a generalized product of $P(G_1)$ and $P(G_2)$.

Keywords: finite groups, direct product, power graphs, product of graphs, isomorphism.

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[3] The second author is partially supported by CSIR-JRF grant.

https://doi.org/10.1016/j.endm.2017.11.014

1571-0653/

1 Introduction

In the last two decades many formulations associating graphs with different algebraic structures are developed viz. zero-divisor graph of rings, idempotent graph of rings, power graph of groups, prime graph of groups, non-zero component graph of vector spaces, subspace intersection graphs, commuting graph of matrices and many others. The directed power graph of a group was defined by Kelarev and Quinn [7]. Then Chakraborty et. al [4] defined the undirected power graph $P(G)$ of a group G as the graph with vertex set G and two distinct vertices a and b are adjacent if either $a^m = b$ or $b^n = a$ for some $m, n \in \mathbb{N}$. There are many articles associating group theoretic behavior of G and the graph theoretic properties of $P(G)$. We refer to the survey [1] for an account of the development on the power graph of groups. Success attained by the researchers in this direction influenced others to generalize power graph of groups to strong power graph [9], deleted power graph $P^*(G)$ [8], enhanced power graph [2], normal subgroup based power graphs [3].

Here we investigate relationship of $P(G_1 \times G_2)$ with $P(G_1)$ and $P(G_2)$. There are three fundamental products of graphs, namely, cartesian product, direct product and strong product of graphs [5]. Here we show that, in general, $P(G_1 \times G_2)$ is not isomorphic to either of these products of $P(G_1)$ and $P(G_2)$. So we introduce a new product of two graphs Γ_1 and Γ_2, which we call generalized product of Γ_1 and Γ_2. Reason behind such naming is that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs.

Our main theorem states that $P(G_1 \times G_2)$ is isomorphic to a generalized product of the power graphs $P(G_1)$ and $P(G_2)$.

2 Main results

We refer to [5] and [10] for the notions on graph theory and to [6] for group theoretic background.

Throughout this article $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}^\sharp = \mathbb{N} \bigcup \{0\}$.

Let us recall different standard notions of products of graphs.

Definition 2.1 Let Γ_1 and Γ_2 be two graphs.

(i) The cartesian product $\Gamma_1 \Box \Gamma_2$ of Γ_1 and Γ_2 is defined as follows:

$$V(\Gamma_1 \Box \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and } (g_1, g_2) \sim (\acute{g_1}, \acute{g_2}) \text{ if and only if either}$$
$$g_1 = \acute{g_1} \text{ and } g_2 \sim \acute{g_2} \text{ or } g_1 \sim \acute{g_1} \text{ and } g_2 = \acute{g_2}.$$