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ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 63 (2017) 245–250
www.elsevier.com/locate/endm

An elementary proof of a conjecture on Graph-Automorphism

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Abstract

In this article, we give an elementary combinatorial proof of a conjecture about the determination of automorphism group of the power graph of finite cyclic groups, proposed by Doostabadi, Erfanian and Jafarzadeh in 2015.

Keywords: group, cyclic group, power graph, degree, automorphism.

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[2] The first author is partially supported by CSIR-JRF grant.

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https://doi.org/10.1016/j.endm.2017.11.020
1571-0653/

1 Introduction

Let G be a finite group. The concept of directed power graph $\overrightarrow{\mathcal{P}(G)}$ was introduced by Kelarev and Quinn [9]. $\overrightarrow{\mathcal{P}(G)}$ is a digraph with vertex set G and for $x, y \in G$, there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Following this Chakrabarty, Ghosh and Sen [4] defined undirected power graph $\mathcal{P}(G)$ of a group G as an undirected graph with vertex set G and two distinct vertices are adjacent if and only if one of them is a positive power of the other.

In 2015, Doostabadi, Erfanian and Jafarzadeh [5] conjectured that for any natural number n,
Aut $(\mathcal{P}(\mathbb{Z}_n)) = (\bigoplus_{d|n, d\neq 1,n} S_{\phi(d)}) \bigoplus S_{\phi(n)+1}$, where ϕ is the Euler's phi function. Although, if n is a prime power, then $\mathcal{P}(\mathbb{Z}_n)$ is complete [4], hence Aut $(\mathcal{P}(\mathbb{Z}_n)) = S_n$. Hence, the conjecture does not hold if $n = p^m$. In 2016, Min Feng, Xuanlong Ma, Kaishun Wang [6] proved that the conjecture holds for the remaining cases, that is for $n \neq p^m$. But their proof uses some what complicated group theoretic arguments. Our aim of this paper is to provide a much more elementary combinatorial proof of the conjecture for $n \neq p^m$ without using any nontrivial group theoretic result.

2 Main Theorem

In this section, first we prove several lemmas and as a consequence, we shall prove the following result, which is the main theorem of this article.

Theorem 2.1 *For $n \neq p^m$ (p prime), Aut $(\mathcal{P}(\mathbb{Z}_n)) = (\bigoplus_{d|n, d\neq 1,n} S_{\phi(d)}) \bigoplus S_{\phi(n)+1}$*

First we prove a technical lemma, which is also the heart of our argument. Before stating it, we have to fix some notations.

Let S be a finite set of positive real numbers. For each subset $B \subseteq S$, let $\prod(B)$ denote the product of all the elements of B. Now let us state and prove the lemma.

Lemma 2.2 *Let $n \geq 2$ and $m_1, m_2, \cdots m_n$ be n positive integers with $m_1 > m_2 > \cdots > m_n$. Let m be any positive integer, and set $A = \{m_1, m_2, \cdots, m_n; m\}$ and $B = \{m_2, m_3, \cdots, m_n\}$. Then to every non empty subset S_B of B, we can associate a proper subset S_A of A, for which $m_1, m \in S_A$ and $\prod(S_B) < \prod(S_A \setminus \{m\})$. The association can be made one to one.*

Proof. We prove this by induction on n. Let $n = 2$. Then $A = \{m_1, m_2, m\}$, $B = \{m_2\}$ and $m_1 > m_2$. The only nonempty subset of B is B itself. To B,