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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 62 (2017) 63–68

www.elsevier.com/locate/endm

# Minimum Linear Arrangements

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## Abstract

Let $G = (V, E)$ be a simple undirected graph. Given distinct labels in $\{1, \cdots, |V|\}$ to the vertices of $G$, we define the weight of an edge $uv \in E$ as the absolute difference between the labels assigned to $u$ and $v$. The minimum linear arrangement problem (MinLA) consists in finding a labeling of the vertices of $G$ such that the sum of the weights of its edges is minimized. It is an NP-Hard problem whose corresponding polyhedron has a factorial number of extreme points. We propose a quadratic model for MinLA and use it to obtain a novel compact mixed integer linear programming (MILP) formulation for the problem, featuring $\mathcal{O}(|V|^2)$ variables and $\mathcal{O}(|V|^2)$ constraints. We show the correctness of the new model and discuss valid inequalities for the problem. Our findings open new insights on the study of effective exact approaches to the problem. Computational experiments show that the new quadratic and mixed linear models performed better than existing ones in the literature for new and benchmark instances of this problem.

*Keywords:* minimum linear arrangement, quadratic programming, mixed integer programming, compact model

https://doi.org/10.1016/j.endm.2017.10.012
1571-0653/

# 1 Introduction

Let $G = (V, E)$ be an undirected simple graph, with $V$ and $E$ being its sets of nodes and edges, respectively. The MinLA problem consists in assigning a permutation $\{\pi_1, \pi_2, \ldots, \pi_{|V|}\}$ of $\{1, 2, \ldots, |V|\}$ to the nodes of $G$, with a one-to-one correspondence, such that the following sum is minimized

$$\sum_{ij \in E} |\pi_i - \pi_j|. \tag{1}$$

This work presents theoretical and practical contributions to the study of MinLA. It concerns a new quadratic model that is used to obtain a compact MILP model for this problem. The importance of these models is that they share an optimal solution structure, i.e., an optimal solution to the quadratic model can be obtained from an optimal one to the MILP model.

We discuss the ideas behind these models and introduce valid inequalities for the MinLA problem. Our model has the smallest number of variables and constraints among implemented and tested models for this problem. There exists a theoretical model to represent the permutahedron [2], based on sorting networks, presenting a small number of variables and constraints to particularly represent the permutahedron. To the best of our knowledge, we are not aware of an implementation of Goemans's work for optimization problems associated with the permutahedron. In fact, our modeling idea works as a sorting tool as in [2].

Numerical experiments performed on new and benchmark instances show that our models outperform known models [1,3] for this problem.

# 2 A nonlinear quadratic formulation

Let us define a directed graph $D = (V, A)$ obtained from $G$ with $A = \{uv \mid u, v \in V,\, u \neq v\}$. Additionally, let us denote by $A(E) \subset A$ the subset of arcs of $D$ where $uv$, $vu \in A(E) \leftrightarrow uv \in E$.

Let $\pi_v \in \mathbb{R}_+^{|V|}$ be continuous variables corresponding to node labels. Let $x_{uv}$, for all $uv \in A$, be a binary variable, associated with the decision of whether or not to include arc $uv$ in the solution, i.e., $x_{uv} = 1$ if arc $uv$ is in the solution, and $x_{uv} = 0$, otherwise. In our model, choosing $x_{uv} = 1$ is equivalent to choosing the label $\pi(v)$ of node $v$ to be larger than the label $\pi(u)$ of node $u$. Finally, let $w_{uv} \in \mathbb{R}_+$, for all $uv \in A$, be continuous variables representing the weight of each arc $uv \in A$. Our model can be written as