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Bispindle in strongly connected digraphs with large chromatic number

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Abstract

A $(k_1 + k_2)$-*bispindle* is the union of k_1 (x, y)-dipaths and k_2 (y, x)-dipaths, all these dipaths being pairwise internally disjoint. Recently, Cohen et al. showed that for every $(2 + 0)$-bispindle B, there exists an integer k such that every strongly connected digraph with chromatic number greater than k contains a subdivision of B. We investigate generalisations of this result by first showing constructions of strongly connected digraphs with large chromatic number without any $(3 + 0)$-bispindle or $(2+2)$-bispindle. Then we show that for any k, there exists γ_k such that every strongly connected digraph with chromatic number greater than γ_k contains a $(2 + 1)$-bispindle with the (y, x)-dipath and one of the (x, y)-dipaths of length at least k.

Keywords: Digraph, chromatic number, subdivision.

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1 Introduction

Throughout this paper, the *chromatic number* of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. In a digraph D, a *directed path*, or *dipath*, is an oriented path where all the arcs are oriented from the initial vertex towards the terminal vertex. A k-*spindle* is the union of k internally disjoint (x, y)-dipaths for some vertices x and y. Vertex x is said to be the *tail* of the spindle and y its *head*. A $(k_1 + k_2)$-*bispindle* is the internally disjoint union of a k_1-spindle with tail x and head y and a k_2-spindle with tail y and head x. In other words, it is the union of k_1 (x, y)-dipaths and k_2 (y, x)-dipaths, all of these dipaths being pairwise internally disjoint.

A classical result due to Gallai, Hasse, Roy and Vitaver is the following.

Theorem 1.1 (Gallai [8], Hasse [9], Roy [11], Vitaver [12]) *If $\chi(D)$ is greater than or equal to k, then D contains a dipath of length $k - 1$.*

This raises the question of which digraphs are subdigraphs of all digraphs with large chromatic number.

A classical theorem by Erdős [6] implies that, if H is a digraph containing a cycle, then there exist digraphs with arbitrarily high chromatic number with no subdigraph isomorphic to H. Thus the only possible candidates to generalise Theorem 1.1 are the *oriented trees* that are orientations of trees. Burr [3] proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k and conjectured an upper bound of $2k - 2$. The best known upper bound, due to Addario-Berry et al. [1], is in $(k/2)^2$.

However the following celebrated theorem of Bondy shows that the story does not stop there.

Theorem 1.2 (Bondy [2]) *Every strongly connected digraph with chromatic number at least k contains a directed cycle of length at least k.*

The strong connectivity assumption is indeed necessary, as transitive tournaments contain no directed cycle but can have arbitrarily high chromatic number.

Observe that a directed cycle of length at least k can be seen as a subdivision of $\vec{C}_k$, the directed cycle of length k. Recall that a *subdivision* of a digraph F is a digraph that can be obtained from F by replacing each arc uv by a dipath from u to v.

Conjecture 1.3 (Cohen et al. [5]) *For every cycle C, there exists a constant $f(C)$ such that every strongly connected digraph with chromatic number at least $f(C)$ contains a subdivision of C.*