Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 62 (2017) 105–110
www.elsevier.com/locate/endm

Computational determination of the largest lattice polytope diameter

Nathan Chadder [1]

Department of Computing and Software
McMaster University
Hamilton, Canada

Antoine Deza [2]

Department of Computing and Software
McMaster University
Hamilton, Canada

Abstract

A lattice (d,k)-polytope is the convex hull of a set of points in dimension d whose coordinates are integers between 0 and k. Let $\delta(d,k)$ be the largest diameter over all lattice (d,k)-polytopes. We develop a computational framework to determine $\delta(d,k)$ for small instances. We show that $\delta(3,4)=7$ and $\delta(3,5)=9$; that is, we verify for $(d,k)=(3,4)$ and $(3,5)$ the conjecture whereby $\delta(d,k)$ is at most $\lfloor (k+1)d/2 \rfloor$ and is achieved, up to translation, by a Minkowski sum of lattice vectors.

Keywords: Lattice polytopes, edge-graph diameter, enumeration algorithm

[1] Email: `chaddens@mcmaster.ca`
[2] Email: `deza@mcmaster.ca`

https://doi.org/10.1016/j.endm.2017.10.019
1571-0653/

1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and if all the vertices are drawn from $\{0, 1, \ldots, k\}^d$, it is referred to as a lattice (d, k)-polytope. Let $\delta(d, k)$ be the largest edge-diameter over all lattice (d, k)-polytopes. Naddef [7] showed in 1989 that $\delta(d, 1) = d$, Kleinschmidt and Onn [6] generalized this result in 1992 showing that $\delta(d, k) \leq kd$. In 2016, Del Pia and Michini [3] strengthened the upper bound to $\delta(d, k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$, and showed that $\delta(d, 2) = \lfloor 3d/2 \rfloor$. Pursuing Del Pia and Michini's approach, Deza and Pournin [5] showed that $\delta(d, k) \leq kd - \lceil 2d/3 \rceil - (k - 3)$ for $k \geq 3$, and that $\delta(4, 3) = 8$. The determination of $\delta(2, k)$ was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that $\delta(d, k) \geq \lfloor (k + 1)d/2 \rfloor$ for all $k \leq 2d - 1$ and proposed Conjecture 1.1.

Conjecture 1.1 *$\delta(d, k) \leq \lfloor (k + 1)d/2 \rfloor$, and $\delta(d, k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors.*

In Section 2, we propose a computational framework which drastically reduces the search space for lattice (d, k)-polytopes achieving a large diameter. Applying this framework to $(d, k) = (3, 4)$ and $(3, 5)$, we determine in Section 3 that $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$.

Theorem 1.2 *Conjecture 1.1 holds for $(d, k) = (3, 4)$ and $(3, 5)$; that is, $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$, and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors.*

Note that Conjecture 1.1 holds for all known values of $\delta(d, k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d, 3) = 2d$. The new entries corresponding to $(d, k) = (3, 4)$ and $(3, 5)$ are entered in bold.