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Advances in Aharoni-Hartman-Hoffman's Conjecture for Split digraphs ⋆

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Abstract

Let k be a positive integer and let D be a digraph. A *(path) k-pack* $\mathcal{P}^k$ of D is a collection of at most k vertex-disjoint paths in D. The *weight* of a k-pack $\mathcal{P}^k$ is the number of vertices covered by it and we say $\mathcal{P}^k$ is *optimal* if its weight is maximum. A vertex-coloring $\mathcal{C}$ is *orthogonal* to a k-pack $\mathcal{P}^k$ if each color class $C \in \mathcal{C}$ meets $\min\{|C|, k\}$ paths of $\mathcal{P}^k$. In 1985, Aharoni, Hartman and Hoffman conjectured that for any optimal k-pack of D there exists a coloring orthogonal to it. In this paper we give a partial answer to this question by presenting two special types of k-packs in split digraphs for which we can always find an orthogonal coloring.

Keywords: path k-pack, vertex-coloring, Aharoni-Hartman-Hoffman's Conjecture

1 Introduction

Throughout this paper, we assume digraphs have neither loops nor parallel arcs (but they may have cycles of length two), paths are directed, and k is a

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fixed positive integer. For terminology not defined here, see [3].

Given a digraph D, we denote its set of vertices by $V(D)$ and its set of arcs by $A(D)$. A *(path) k-pack* $\mathcal{P}^k$ is a collection of at most k (vertex-)disjoint paths of a digraph D. The *weight* of $\mathcal{P}^k$ is defined as $|\cup_{P\in\mathcal{P}^k} V(P)|$ and denoted by $||\mathcal{P}^k||$. A k-pack is *optimal* if its weight is maximum. A (vertex-)coloring $\mathcal{C}$ is *orthogonal* to a k-pack $\mathcal{P}^k$ if each $C \in \mathcal{C}$ meets $\min\{|C|, k\}$ paths of $\mathcal{P}^k$. Aharoni, Hartman and Hoffman [1] proposed the following conjecture.

Conjecture 1.1 (Aharoni, Hartman and Hoffman, 1985) *Let $\mathcal{P}^k$ be an optimal k-pack of a digraph D. There exists a coloring of D orthogonal to $\mathcal{P}^k$.*

This conjecture remains open, but we know it holds for $k = 1$ [4,7], for $k \geq \pi(D)$ [5] (where $\pi(D)$ is the cardinality of a minimum path partition of D), when the optimal k-pack has at least one trivial path [5], for bipartite digraphs [5], and for acyclic digraphs [1]. Conjecture 1.1 is a generalization [5] of the following conjecture proposed by Linial [6]. The *k-norm* of a coloring $\mathcal{C}$, denoted by $|\mathcal{C}|_k$, is defined as $\sum_{C\in\mathcal{C}} \min\{|C|, k\}$. For a digraph D, we denote the smallest k-norm of a coloring for D by $\chi_k(D)$ and the weight of an optimal k-pack for D by $\lambda_k(D)$.

Conjecture 1.2 (Linial, 1981) *For any digraph D, $\chi_k(D) \leq \lambda_k(D)$.*

Since Conjecture 1.1 is a generalization of Conjecture 1.2, the latter holds for every case in which the former holds. In addition, we know Conjecture 1.2 also holds for split digraphs [5] which we define next.

A *semi-complete digraph* is a digraph D such that for every pair of distinct vertices u and v, $uv \in A(D)$ or $vu \in A(D)$. A digraph D is *split* if there exists a partition $\{X, Y\}$ of $V(D)$ such that $D[X]$ is a semi-complete digraph and Y is a stable set of D. We denote such a split digraph by $D[X, Y]$.

So far, the single case for which Conjecture 1.2 holds and it is unknown whether Conjecture 1.1 is valid is for split digraphs. This observation teases a curious mind and raises the natural question as to whether it would be possible to prove the Conjecture 1.2 for split digraphs. In this paper we give a partial answer to this question by presenting two special types of k-packs in split digraphs for which we can always find an orthogonal coloring.

Let $D[X, Y]$ be a split digraph and let $\mathcal{P}^k$ be a k-pack of D. We define $V(\mathcal{P}^k)$ as the set $\cup_{P\in\mathcal{P}^k} V(P)$ of vertices *covered* by $\mathcal{P}^k$ and $\overline{V}(\mathcal{P}^k)$ as the set $V(D)\backslash V(\mathcal{P}^k)$ of vertices *not covered* by $\mathcal{P}^k$. We say a k-pack is *Y-covering* if $V(\mathcal{P}^k) \supseteq Y$ and *X-covering* if $V(\mathcal{P}^k) \supseteq X$. In Section 2 we prove that Conjecture 1.1 holds for *Y-covering* optimal k-packs in general and in Section 3 we prove it holds for *X-covering* optimal k-packs with a particular property.