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Inapproximability Ratios for Crossing Number

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Abstract

Assuming $P \neq NP$, we show that if there is a constant-factor polynomial c-approximation algorithm for Crossing Number, then $c \geq 2 - \frac{16}{17} \approx 1.058824$. Adding the Unique Games Conjecture to the hypotheses, then $c \geq 2 - \alpha \approx 1.121433$, where α is the approximation ratio of the algorithm for Maximum Cut by Goemans and Williamson.

Keywords: crossing number, computational complexity, approximation, inapproximability

1 Introduction

A drawing of a graph G is a function $D : G \to \mathbb{R}^2$ that takes each vertex to a distinct point, and each edge $uv \in E(G)$ to an arc connecting $D(u)$ to $D(v)$ that does not contain the image of any other vertex. A crossing occurs whenever the image of two edges coincide somewhere other than their extremes. The crossing number $\mathrm{cr}(D)$ of D is the sum, for all pairs e, e' of edges, of the crossings between $D(e)$ and $D(e')$. When the edges e and e'

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have weights $w(e)$ and $w(e')$, each crossing counts $w(e)w(e')$ to the sum. The minimum crossing number of all drawings of G is denoted by $\mathrm{cr}(G)$.

The Crossing Number problem consists in finding a drawing of a graph G whose crossing number equals $\mathrm{cr}(G)$. Garey and Johnson [3] showed it to be NP-hard, followed by Hliněný [6], Cabello and Mohar [2], who restricted the result to some specific classes of graphs. Cabello [1] showed there is a constant $c > 1$ such that there is no polynomial c-approximation for Crossing Number, given the assumption that $P \neq NP$, by a reduction from the Multiway Cut problem. This paper presents a reduction from Maximum Cut that implies some precise lower bounds for such a constant.

2 Reduction

We will use weighted graphs in the construction to make it simpler. The weights will always be positive integers. Despite the use of weighted graphs, the results here also apply to the unweighted case, since each edge weighting w may be substituted by w parallel paths. Conversion between the two cases may be done by drawing the paths close to the weighted-edge arc, and conversely by drawing the weighted edge by the path with fewer crossings. From now on, unitary-weight edges will be called *light*, while the others will be called *heavy*.

In order to find an inapproximability ratio for Crossing Number, we show an approximation algorithm for the Maximum Cut of a graph G based on minimizing the crossings on a related graph G'. Given a graph G with n vertices and m edges, we construct the graph G' as follows:

- For each vertex $v \in V(G)$, create the vertices x_v, y_v, z_v^a, z_v^b and z_v^c. Besides, create the edges $z_v^a z_v^b$ and $z_v^b z_v^c$, along with $x_v z_v^i$ and $y_v z_v^i$ for all $i \in \{a, b, c\}$, all weighting m^3 (Fig. 1).
- Let $v_1, v_2, \ldots, v_n$ be any ordering of $V(G)$. Add the edges $z_{v_1}^c z_{v_2}^a$, $z_{v_2}^c z_{v_3}^a$, $\ldots$, $z_{v_{n-1}}^c z_{v_n}^a$ and $z_{v_n}^c z_{v_1}^a$, all weighting m^3.
- For each edge $uv \in E(G)$, add to G' the edges $x_u y_v$ and $x_v y_u$, with unitary weight.

Since G' has $5n$ vertices and $2m + 9n$ edges, it may be constructed in polynomial time on the size of G. Moreover, the unweighted counterpart of G', where the weighted edges are replaced by paths, may also be constructed in polynomial time, because the weights are all polynomial on the size of G.

Consider the subgraph of G' induced only by the heavy edges, and let us study its planar drawings. First, note that the vertices z_v^i, for all $v \in V(G)$ and $i \in \{a, b, c, \}$, induce a circle, which we will call *main circle*. For each