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Gallai's conjecture for graphs with treewidth 3

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Abstract

Gallai conjectured (1966) that the edge-set of a simple graph G with n vertices can be covered by at most $(n+1)/2$ edge-disjoint paths. Lovász (1968) verified this conjecture for graphs with at most one vertex of even degree, and Pyber (1996) verified it for graphs in which every cycle contains a vertex of odd degree. Recently, Bonamy and Perrett verified this Conjecture for graphs with maximum degree at most 5. In this paper, we verify the Conjecture for graphs with treewidth at most 3.

Keywords: Gallai's conjecture, path decomposition, treewidth, reducing subgraph

1 Introduction

A *path decomposition* $\mathcal{D}$ of a graph G is a set $\{H_1, \ldots, H_k\}$ of edge-disjoint paths of G that cover the edge-set of G. The size of a minimum path decomposition of G is called the *path number* of G, and is denoted by $\mathrm{pn}(G)$. In this paper, we focus on the following conjecture concerning minimum path decompositions of graphs (see [4,8]).

Conjecture 1.1 (Gallai, 1966) *Every graph on n vertices admits a path decomposition of size at most $\lfloor (n+1)/2 \rfloor$.*

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In 1968, Lovász [8] proved that a graph with n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles. A consequence of this result is that if G is a graph with at most one vertex of even degree, then $\mathrm{pn}(G) = \lfloor n/2 \rfloor$. Pyber [9] improved this result, verifying Gallai's conjecture for graphs in which every cycle has a vertex of odd degree.

Theorem 1.2 (Lovász, 1968) *If G is a graph with n vertices, then G can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles.*

Theorem 1.3 (Pyber, 1996) *If G is a graph with n vertices in which every cycle contains a vertex of odd degree, then $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$.*

In 2005, Fan [6] further improved this result, but the conjecture is still open. In another direction, Conjecture 1.1 was verified for a family of even regular graphs [5], and for a family of triangle-free graphs [7]. Bonamy and Perrett [3] verified Conjecture 1.1 for graphs with maximum degree at most 5, showing that in a minimal (on the number of vertices) counter-example every cycle contains a vertex of odd degree, in contradiction to Theorem 1.3.

In this paper, we verify Gallai's conjecture for graphs with treewidth at most 3. Our technique consists in, given a minimal (on the number of vertices) counter-example G, finding a subgraph H of G such that $G' = G - E(H)$ contains at most $|V(G)| - 2r$ non-isolated vertices, and $\mathrm{pn}(H) \leq r$. Thus, G' is not a counter-example. Moreover, we show how to obtain H in such a way that no component of G' has path number $(n+1)/2$. Therefore, the decomposition $\mathcal{D}$ of G obtained by joining minimum path decompositions of H and G' is such that $|\mathcal{D}| \leq \lfloor n/2 \rfloor$. The graph H is called an *r-reducing subgraph*, and is discussed in Section 2. As a byproduct of our main result (Theorem 3.2), the only graphs with treewidth at most 3 and path number $(n+1)/2$ are isomorphic to K_3 and K_5^- (the graph obtained from K_5 by removing exactly one edge).

This work is organized as follows. In Section 2, we define reducing subgraphs and present some technical lemmas. In Section 3, we verify Conjecture 1.1 for graphs with treewidth at most 3. In Section 4, we study a conjecture similar to Conjecture 1.1. In Section 5, we give some concluding remarks. Due to space limitations, we present only the sketch of some proofs.

2 Reducing subgraphs

Let G be a graph and let H be a subgraph of G. Given a positive integer r, we say that H is an *r-reducing subgraph of* G if $G - E(H)$ has at least $2r$