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On the (di)graphs with (directed) proper connection number two [1]

Guillaume Ducoffe [a,2] Ruxandra Marinescu-Ghemeci [b,c]
Alexandru Popa [a,b]

[a] *National Institute for Research and Development in Informatics, Romania*

[b] *University of Bucharest, Faculty of Mathematics and Computer Science*

[c] *The Research Institute of the University of Bucharest ICUB, Romania*

Abstract

A coloring of a graph G is *properly connected* if every two vertices of G are the ends of a properly colored path. We study the *complexity* of computing the *proper connection number* (minimum number of colors in a properly connected coloring) for edge and vertex colorings, in undirected and directed graphs, respectively. First we disprove some conjectures of Magnant et al. (2016) on characterizing the strong digraphs with *proper arc connection number* at most two. Then, we prove that deciding whether a given digraph has proper arc connection number at most two is NP-complete. We initiate the study of proper vertex connectivity in digraphs and we prove similar results as for the arc version. Finally, we present polynomial-time recognition algorithms for *bounded-treewidth* graphs and *bipartite* graphs with *proper edge connection number* at most two.

Keywords: proper connection; digraphs; bipartite; even dicycles; NP-complete.

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[2] Email: guillaume.ducoffe@ici.ro

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1 Introduction

We study a relaxed variant of proper colorings, introduced by Borozan et al. [2], where we only impose for every two vertices u, v to have a properly colored (di)path from u to v—the (di)graph itself may not be properly colored. The latter concept is sometimes called *proper connectivity*. Properly colored paths have applications in many fields like genetics [7] or communication networks. As an example, it is desirable in wireless networks to have all the parties connected and to avoid interference by ensuring that the incoming and the outgoing signal from a tower should be on different frequencies. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and a color to each edge corresponding to the frequency used for the communication. Then, the number of frequencies needed to assign the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Related work. The proper connection number in undirected graphs (for *edge* colorings) was first defined in [2] by Borozan et al., where they relate the proper connection number with the graph connectivity. Since then the problem was intensively studied from the combinatorial point of view [1,2,9,15,13]. In particular, bounds on the proper connection number of random graphs and bipartite graphs have been proved, respectively, in [9] and in [2,11,10]. Relationships between proper connection number and domination number can be found in [13]. Many generalizations of proper connectivity have been proposed [2,11,1,14]. For instance, in this paper, we also study some notions of *vertex proper connection*, *i.e.*, vertex-coloring versions of the proper connection number (see [6]). More recently, Magnant et al. studied the proper *arc* connection number for strong digraphs [15]. They proved that this number is always at most three and they asked to characterize the digraphs with proper connection number at most two. In particular, they conjectured that all such digraphs must contain an even dicycle. For more details, see the following survey on the proper edge connection number: [12]. Nevertheless, no complexity results have been proved for proper connectivity, until our work. Our goal is to fill in this gap in the literature.

Proper connectivity is related to the rainbow connectivity, defined in [5]. Since computing the rainbow connection number is NP-hard and not FPT for any fixed $k \geq 2$ [3] it is natural to study the complexity of computing the proper edge connectivity – which can be seen a relaxed variant of rainbow connectivity.