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Reducing the Chromatic Number by Vertex or Edge Deletions

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Abstract

A vertex or an edge in a graph is critical if its deletion reduces the chromatic number of the graph by one. We consider the problems of testing whether a graph has a critical vertex or a critical edge, respectively. We give a complete classification of the complexity of both problems for H-free graphs, that is, graphs with no induced subgraph isomorphic to H. Moreover, we show that an edge is critical if and only if its contraction reduces the chromatic number by one. Hence, we obtain the same classification for the problem of testing if a graph has an edge whose contraction reduces the chromatic number by one. As a consequence of our results, we are also able to complete the complexity classification of the more general vertex deletion and edge contraction blocker problems for H-free graphs when the graph parameter is the chromatic number.

Keywords: edge contraction, vertex deletion, chromatic number.

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1 Introduction

A vertex or an edge of a graph G is *critical* if its removal reduces the chromatic number $\chi(G)$ by one. An edge is *contraction-critical* if its contraction reduces $\chi(G)$ by one. We call the problems of deciding if a graph has a critical vertex, a critical edge or a contraction-critical edge Critical Vertex, Critical Edge and Contraction-Critical Edge, respectively. It is not difficult to show that these three problems are computationally hard in general. Here we classify their computational complexity for graphs with no induced subgraph isomorphic to some specified graph H; we call such graphs H*-free*.

Each of the above decision problems can be generalized as follows. Let S be a fixed set of one or more graph operations, and let π be some fixed graph parameter. Given a graph G, and an integer k, we ask if G can be modified into a graph G' by using at most k operations from S so that $\pi(G') \leq \pi(G) - d$ for some given *threshold* $d \geq 0$. Such problems are called *blocker problems*, as the vertices or edges involved "block" some desirable graph property (such as being colorable with only a few colors). Over the last few years, blocker problems have been well studied, see for instance [1,2,3,4,5,9,10,11,12]. In these papers, the set S consists of a single operation that is either a vertex deletion vd, an edge deletion ed, or an edge contraction ec. The decision problems are called Vertex Deletion Blocker(π) if $S = \{\mathsf{vd}\}$, Edge Deletion Blocker(π) if $S = \{\mathsf{ed}\}$ and Contraction Blocker(π) if $S = \{\mathsf{ec}\}$. By taking $d = k = 1$ and $\pi = \chi$ we obtain the problems Critical Vertex, Critical Edge and Contraction-Critical Edge. The complexities of Vertex Deletion Blocker(χ) and Contraction Blocker(χ) are known for H-free graphs if H is connected [10]. As a consequence of our results for $k = d = 1$, we can complete these two classifications for all graphs H, just as we did in a previous paper [11] for $\pi = \alpha$ (independence number) and $\pi = \omega$ (clique number), except for the case when $\pi = \omega$, $S = \{\mathsf{ec}\}$ and $H = C_3 + P_1$. The Edge Deletion Blocker(χ) problem is known [1] to be polynomial-time solvable for threshold graphs and NP-hard for cobipartite graphs. For this problem we obtain a partial classification that leaves exactly two cases open.

Terminology. The graph $G + G'$ is the disjoint union of the graphs G and G'. The graph pG is the disjoint union of p copies of G. We let K_n, P_n and C_n be the complete graph, path and cycle on n vertices, respectively. For a subset $S \subseteq V$ of a graph G, we let $G[S]$ be the subgraph of G *induced* by S. We write $H \subseteq_i G$ if H is an induced subgraph of G. The graph $\overline{G}$ is the complement of G. A graph G is $(H_1, \ldots, H_p)$*-free* if G is H-free for every $H \in \{H_1, \ldots, H_p\}$. The contraction of an edge uv removes u and v from V and replaces them by