Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 62 (2017) 285–290 www.elsevier.com/locate/endm

On the recognition of neighborhood inclusion posets

Pablo De Caria [1]

CONICET, Departamento de Matemática, UNLP, Argentina

Abstract

Let G be a simple graph. When we order the different closed neighborhoods of G by inclusion, the resulting poset is called the neighborhood inclusion poset. In this paper, we show that the problem of determining whether a poset is a neighborhood inclusion poset is NP-complete. We also apply this result to prove the NP-completeness of another problem about clique trees of chordal graphs and compatible trees of dually chordal graphs.

Keywords: Poset, neighborhood inclusion, chordal graph, clique tree, dually chordal graph, compatible tree

1 Introduction

A *poset* P is a pair $(X, \leq)$ where X is a set and $\leq$ is a partial order (a reflexive, antisymmetric and transitive relation) defined on X. Every poset can be modelled through the inclusion of sets. In fact, if we make every $x \in X$ correspond to the set $D_x = \{y \in X : \ y \leq x\}$, then it holds that $u \leq v$ if and only if $D_u \subseteq D_v$, for every pair u, v of elements of X.

[1] Email: `pdecaria@mate.unlp.edu.ar`

https://doi.org/10.1016/j.endm.2017.10.049
1571-0653/

Many special posets have been studied, like for example the posets that can be modelled through the inclusion of intervals of the real line [6,7]. In this paper, we will consider the inclusion order of the neighborhoods of a graph.

Let G be a graph. The closed neighborhood $N[v]$ of a vertex v of G consists of v and of all the vertices adjacent to it. Denote the set of all the different closed neighborhoods of G by $\mathcal{N}_G$. The *neighborhood inclusion poset* of G is defined to be the pair $(\mathcal{N}_G, \subseteq)$ consisting of the different closed neighborhoods of G ordered by inclusion. We say that a poset P is a *neighborhood inclusion* poset if there exists a graph G such that $(\mathcal{N}_G, \subseteq)$ is isomorphic to P.

Fig. 1. A graph and the Hasse diagram of its neighborhood inclusion poset.

The structure and recognition of neighborhood inclusion orders in graphs has already been considered previously, but restricted to some particular classes of graphs [1]. In this paper, we consider the problem of determining whether a given poset is the neighborhood inclusion poset of some graph and we show that it is NP-complete by establishing a connection with the Set Basis problem.

This proof was motivated by another problem about chordal and dually chordal graphs, i.e., clique graphs of some chordal graph. Both classes can be characterized by the existence of tree representations. A *clique tree* of a graph G is a tree T whose vertex set is the set $\mathcal{C}(G)$ of maximal cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of maximal cliques of G that contain v induces a subtree in T. A graph is chordal if and only if it has a clique tree [8]. A *compatible tree* of a graph G is a tree T that has the same vertices as G and such that every maximal clique and every closed neighborhood of G induces a subtree in G. A graph is dually chordal if and only if it has a compatible tree [2].

Given a family $\mathcal{T}$ of trees on a vertex set V, the problem of determining whether $\mathcal{T}$ is the family of all clique trees of some chordal graph G can be solved in polynomial time [4]. We also know that every family of compatible trees of a dually chordal graph is also the family of clique trees of some chordal graph [5]. However, it is shown in [3] that not every family of clique trees of a chordal graph is the family of compatible trees of some dually chordal graph, leaving it an open problem to determine the complexity of recognizing a family