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On graphs with a single large Laplacian eigenvalue

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Abstract

We address the problem of characterizing those graphs G having only one Laplacian eigenvalue greater than or equal to the average degree of G. Our conjecture is that these graphs are stars plus a (possible empty) set of isolated vertices.

Keywords: anticomponents, Laplacian eigenvalues, stars

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1 Introduction

Let G be a graph on n vertices and m edges and let $d_1 \geq \cdots \geq d_n$ be its degree sequence. Let $A(G)$ be its adjacency matrix and $D(G)$ its diagonal matrix of vertex degrees. The *Laplacian matrix* of G is the positive semidefinite matrix $L(G) = D(G) - A(G)$. The spectrum of $L(G)$ is called the *Laplacian spectrum* of G and is denoted by $Lspec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, where $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Understanding the distribution of Laplacian eigenvalues of graphs is a problem that is both relevant and difficult. It is relevant due to the many applications related to Laplacian matrices (see, for example [8,9]). It seems to be difficult because little is known about how the n Laplacian eigenvalues are distributed in the interval $[0, n]$.

Our main motivation is understanding the structure of graphs that have few large Laplacian eigenvalues. In particular, we would like to characterize graphs that have a single large Laplacian eigenvalue. What do we mean by a large Laplacian eigenvalue? A reasonable measure is to compare this eigenvalue with the average of all eigenvalues. Since the average of Laplacian eigenvalues equals the average degree $\overline{d}(G) = \frac{2m}{n}$ of G, we say that *a Laplacian eigenvalue is large if it is greater than or equal to the average degree.*

Inspired by this idea, the paper [2] introduces the spectral parameter $\sigma(G)$ which counts the number of Laplacian eigenvalues greater than or equal to $\overline{d}(G)$. Equivalently, $\sigma(G)$ is the largest index i for which $\mu_i \geq \frac{2m}{n}$.

There is evidence that $\sigma(G)$ plays an important role in defining structural properties of a graph G. For example, it is related to the clique number ω of G (the number of vertices of the largest induced complete subgraph of G) and it also gives insight about the Laplacian energy of a graph [10,2]. Moreover several structural properties of a graph are related to σ (see, for example [1,2]).

In this paper we are concerned with furthering the study of $\sigma(G)$. In particular, we deal with a problem posed in [2] which asks for characterizing all graphs having $\sigma(G) = 1$, *i.e.* having only one large Laplacian eigenvalue. We conjecture that these graphs are some stars plus a (possible empty) set of isolated vertices ($K_{1,r}$ denotes the star on $r+1$ vertices and $+$ the disjoint union):

Conjecture 1.1 *Let G be a graph. Then $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s \geq 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 \leq s < r - 1$.*

In this work, we show that this conjecture is true if it holds for graphs which are simultaneously connected and co-connected (Conjecture 4.3) and prove that Conjecture 1.1 is true for forests and extended P_4-laden graphs [4]