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Approximating the cone of copositive kernels to estimate the stability number of infinite graphs

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Abstract

It has been shown that the stable set problem in an infinite compact graph, and particularly the kissing number problem, reduces to an optimization problem over the cone of copositive kernels. We propose two converging hierarchies approximating this cone. Both are extensions of existing inner hierarchies for the finite dimensional copositive cone. We implement the first two levels of the new hierarchies for the kissing number problem.

Keywords: copositive programming, semidefinite approximations, lifting, kissing number

1 Introduction

Consider the stable set problem in an infinite dimensional undirected graph $G = (V, E)$. This problem is motivated by fundamental combinatorial optimization arrangements, such as sphere packings, convex body packings, binary codes and spherical codes [5]. The *kissing number* problem, for which

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we implement our hierarchies, is an instance of the spherical codes problem. Following de Laat and Vallentin [6], we work with *compact topological packing graphs*, graphs whose vertex set is compact and every finite clique is contained in an open clique. The stability number of these graphs is finite.

Let $V \subset \mathbb{R}^n$ be a compact set. Denote the space of real-valued continuous functions on V by $C(V)$. We call *kernels* the following subset of continuous functions: $\mathcal{K}(V) = \{K \in C(V \times V) : K(x,y) = K(y,x),\ \forall\, x,y \in V\}$. A kernel K is said to be *copositive* if any finite principal submatrix of K is copositive [4]. We denote copositive kernels on V by $\mathcal{COP}(V)$. The stability number of $G = (V,E)$ is the optimal solution to the following problem [4]:

$$\begin{aligned} \alpha(G) = \inf_{K\in\mathcal{K}(V),\ \lambda\in\mathbb{R}} \quad & \lambda \\ \text{s.t.} \quad & K \in \mathcal{COP}(V) \\ & K(v,v) = \lambda - 1 && \text{for all } v \in V \\ & K(u,v) = -1 && \text{for all } (u,v) \notin E. \end{aligned} \tag{1}$$

Optimization over $\mathcal{COP}(V)$ is NP-hard even for finite V, so the goal is to replace $\mathcal{COP}(V)$ by simpler convex objects and obtain some bounds on $\alpha(G)$.

In this paper we propose *inner* approximations for $\mathcal{COP}(V)$. Let $\mathbb{S}^n$ be the space of $n \times n$ symmetric matrices over $\mathbb{R}$, and denote the set $\{1, ..., n\}$ by $[n]$. Kernels generalize the notion of symmetric matrices since $\mathbb{S}^n \cong \mathcal{K}([n])$. The following inner hierarchies were introduced for $\mathcal{COP}([n])$ by Parrilo [8], Peña et al. [9] and De Klerk and Pasechnik [2] respectively:

$$\mathcal{K}_r^n = \Big\{M \in \mathbb{S}^n : \left(\sum_{i=1}^n x_i^2\right)^r \sum_{i=1}^n \sum_{j=1}^n M_{ij} x_i^2 x_j^2 \text{ is a sum of squares}\Big\}. \tag{2}$$

$$\begin{aligned} Q_r^n = \Big\{M \in \mathbb{S}^n : (e^\top x)^r (x^\top M x) = \sum_{|\beta|=r} x^\beta x^\top N_\beta x + \sum_{|\beta|=r} x^\beta x^\top S_\beta x, \\ N_\beta, S_\beta \in \mathbb{S}^n,\ N_\beta \geq 0 \text{ and } S_\beta \succeq 0 \text{ for all } \beta \in \mathbb{N}^n, |\beta| = r\Big\}, \end{aligned} \tag{3}$$

$$\mathcal{C}_r^n = \Big\{M \in \mathbb{S}^n : (e^\top x)^r (x^\top M x) \text{ has nonnegative coefficients}\Big\}, \tag{4}$$

where $e = (1, ..., 1)$, $|\beta| := \beta_1 + ... + \beta_n$ and $x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$.

We have the inclusions $\mathcal{C}_r^n \subseteq Q_r^n \subseteq \mathcal{K}_r^n \subseteq \mathcal{COP}([n])$ for any r [9], and all hierarchies converge to $\mathcal{COP}([n])$ as r grows to infinity. We generalize the sets (4) and (3) to the case of copositive kernels on any compact $V \subset \mathbb{R}^n$ and show convergence of the obtained hierarchies.

Some approximations for the cone of copositive kernels on general V already exist. The first one comes from replacing copositivity in the definition