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The extremal functions for triangle-free graphs with excluded minors

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ABSTRACT

We prove two results:

1. A graph G on at least seven vertices with a vertex v such that $G-v$ is planar and t triangles satisfies $|E(G)| \leq 3|V(G)| - 9 + t/3$.
2. For $p = 2, 3, \ldots, 9$, a triangle-free graph G on at least $2p-5$ vertices with no K_p-minor satisfies $|E(G)| \leq (p-2)|V(G)| - (p-2)^2$.

1. Introduction

All *graphs* in this paper are finite and simple. *Cycles* have no "repeated" vertices. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An *H-minor* is a minor isomorphic to H. Mader [6] proved the following beautiful theorem.

Theorem 1.1. *For $p = 2, 3, \ldots, 7$, a graph with no K_p-minor and $V \geq p-1$ vertices has at most $(p-2)V - \binom{p-1}{2}$ edges.*

For large p however, a graph on V vertices with no K_p-minor can have up to $\Omega(p\sqrt{\log p}V)$ edges as shown by several people (Kostochka [4,5], and Fernandez de la Vega [2] based on Bollobás, Catlin and Erdös [1]). Already for $p = 8, 9$, there are K_p-minor-free graphs on V vertices with strictly more than $(p-2)V - \binom{p-1}{2}$ edges, but the exceptions are known. Given a graph G and a positive integer k, we define (G, k)-*cockades* recursively as follows. A graph isomorphic to G is a (G, k)-cockade. Moreover, any graph isomorphic to one obtained by identifying complete subgraphs of size k of two (G, k)-cockades is also a (G, k)-cockade, and every (G, k)-cockade is obtained this way. The following is a theorem of Jørgensen [3].

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Theorem 1.2. *A graph on* $V \geq 7$ *vertices with no* K_8*-minor has at most* $6V - 21$ *edges, unless it is a* $(K_{2,2,2,2,2}, 5)$*-cockade.*

The next theorem is due to Song and the first author [10].

Theorem 1.3. *A graph on* $V \geq 8$ *vertices with no* K_9*-minor has at most* $7V - 28$ *edges, unless it is a* $(K_{1,2,2,2,2,2}, 6)$*-cockade or isomorphic to* $K_{2,2,2,3,3}$.

The first author and Zhu [11] conjecture the following generalization.

Conjecture 1.4. *A graph on* $V \geq 9$ *vertices with no* K_{10}*-minor has at most* $8V - 36$ *edges, unless it is isomorphic to one of the following graphs:*

(1) *a* $(K_{1,1,2,2,2,2,2}, 7)$*-cockade,*
(2) $K_{1,2,2,2,3,3}$,
(3) $K_{2,2,2,2,2,3}$,
(4) $K_{2,2,2,2,2,3}$ *with an edge deleted,*
(5) $K_{2,3,3,3,3}$,
(6) $K_{2,3,3,3,3}$ *with an edge deleted,*
(7) $K_{2,2,3,3,4}$, *and*
(8) *the graph obtained from the disjoint union of* $K_{2,2,2,2}$ *and* C_5 *by adding all edges joining them.*

McCarty and the first author studied the extremal functions for *linklessly embeddable graphs*: graphs embeddable in 3-space such that no two disjoint cycles form a non-trivial link. Robertson, Seymour, and the first author [9] showed that a graph is linklessly embeddable if and only if it has no minor isomorphic to a graph in the *Petersen family*, which consists of the seven graphs (including the Petersen graph) that can be obtained from K_6 by ΔY- or $Y\Delta$-transformations. Thus, Mader's theorem implies that a linklessly embeddable graph on V vertices has at most $4V - 10$ edges. McCarty and the first author [8] proved the following.

Theorem 1.5. *A bipartite linklessly embeddable graph on* $V \geq 5$ *vertices has at most* $3V - 10$ *edges, unless it is isomorphic to* $K_{3,V-3}$.

In the same paper McCarty and the first author made the following three conjectures.

Conjecture 1.6. *A triangle-free linklessly embeddable graph on* $V \geq 5$ *vertices has at most* $3V - 10$ *edges, unless it is isomorphic to* $K_{3,V-3}$.

As a possible approach to Conjecture 1.6 McCarty and the first author proposed the following.

Conjecture 1.7. *A linklessly embeddable graph on* $V \geq 7$ *vertices with* t *triangles has at most* $3V - 9 + t/3$ *edges.*

The third conjecture of McCarty and the first author is as follows.

Conjecture 1.8. *For* $p = 2, 3, \ldots, 8$, *a bipartite graph on* $V \geq 2p - 5$ *vertices with no* K_p*-minor has at most* $(p-2)V - (p-2)^2$ *edges.*

1.1. Our results

We first give a partial result to Conjectures 1.6 and 1.7. An *apex graph* is a graph G with a vertex a such that $G - a$ is planar. All apex graphs are linklessly embeddable. We show that Conjectures 1.6 and 1.7 hold for apex graphs:

Theorem 1.9. *A triangle-free apex graph on* $V \geq 5$ *vertices has at most* $3V - 10$ *edges, unless it is isomorphic to* $K_{3,V-3}$*. Moreover, an apex graph on* $V \geq 7$ *vertices with* t *triangles has at most* $3V - 9 + t/3$ *edges.*