European Journal of Combinatorics 74 (2018) 11–26

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Enumerations relating braid and commutation classes

Susanna Fishel [a], Elizabeth Milićević [b], Rebecca Patrias [c], Bridget Eileen Tenner [d]

[a] School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ, USA
[b] Department of Mathematics & Statistics, Haverford College, Haverford, PA, USA
[c] LaCIM, Université du Québec à Montréal, Montréal (Québec), Canada
[d] Department of Mathematical Sciences, DePaul University, Chicago, IL, USA

ARTICLE INFO

Article history:
Received 24 August 2017
Accepted 14 July 2018

ABSTRACT

We obtain an upper and lower bound for the number of reduced words for a permutation in terms of the number of braid classes and the number of commutation classes of the permutation. We classify the permutations that achieve each of these bounds, and enumerate both cases.

1. Introduction

The symmetric group S_n is the Coxeter group generated by the adjacent transpositions $\{s_1, \ldots, s_{n-1}\}$. Every $w \in S_n$ can be expressed as a minimal-length product of these generators, and a given permutation may have more than one such representation. These representations are *reduced expressions* for w, each of which can be encoded as a *reduced word* for w. The set of reduced words $R(w)$ possesses a rich combinatorial structure that has been studied from many different perspectives. For example, Stanley showed that the number $|R(w)|$ of reduced words for w can be calculated in terms of Young tableaux of particular shapes [18], but this is not an easy value to calculate outside of special cases. Another common technique explores various quotients of $R(w)$ under the relations governing the adjacent transpositions. For example, the reduced words $\mathbf{u}$ and $\mathbf{v}$ for $w \in S_n$ are in the same *commutation class* if we can obtain $\mathbf{u}$ from $\mathbf{v}$ by applying a sequence of commutation relations of the form $s_is_j = s_js_i$ for $|i-j| > 1$. The *braid classes* of w are defined similarly, in terms of the braid relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. In this paper, we give upper and lower bounds for $|R(w)|$, in terms of the number of braid and commutation classes of the permutation w.

E-mail addresses: sfishel1@asu.edu (S. Fishel), emilicevic@haverford.edu (E. Milićević), patriasr@lacim.ca (R. Patrias), bridget@math.depaul.edu (B.E. Tenner).

https://doi.org/10.1016/j.ejc.2018.07.002

Quotients of $R(w)$ under these Coxeter relations have been studied before, but almost all of that work has focused on the commutation classes, with very little attention paid to the braid classes. In [6], Elnitsky presented a bijection from the commutation classes of w to rhombic tilings of a particular polygon that depends on w, which Tenner utilized extensively in [22]. Jonsson and Welker [10] considered certain cell complexes where the commutation classes appear. Bergeron, Ceballos, and Labbé [3] studied a similar scenario, but their work does consider quotienting out by more general Coxeter relations. Meng [13] studied both the number of commutation classes and their network relationships, and Bédard developed recursive formulas for the number of reduced words in each commutation class in [2], as well as more detailed statistics for the case of the longest permutation w_0. Such focus on certain fixed elements has also been a productive approach to studying the set of reduced words. For example, the higher Bruhat order $B(n, 2)$, studied by Manin and Schechtman [11] and Ziegler [24] is a partial order on the commutation classes of w_0.

Another important family of permutations for which the study of the commutation classes has been especially fruitful are the fully commutative elements, which are those having a single commutation class. In their study of Schubert polynomials, Billey, Jockusch, and Stanley [4] showed that fully commutative permutations are 321-avoiding. Stembridge investigated and enumerated the fully commutative elements in all Coxeter groups in [19–21], and he provided further examples of their appearance in algebra. More recently, Green and Losonczy [8] and Hanusa and Jones [9] have worked on adaptations of full commutativity to root systems and affine permutations, respectively.

The equivalence classes of $R(w)$ under the braid relations, on the other hand, have been noticeably less well studied, both in terms of how they partition the collection of reduced words, and as objects of structure themselves. Zollinger [25] used an encoding of the reduced words to provide formulas for the size of the braid classes, and the previously cited work in [3] showed that the graph on braid classes, with edges indicating when an element of one class can be transformed into an element of the other class by a commutation move, is bipartite.

As with commutation classes, focusing on a particular element in S_n, such as the longest permutation w_0, has been a fruitful approach to understanding some of the relevance and influence of braid moves in reduced words. Reiner and Roichman [15] defined a graph whose vertices are the elements of $R(w_0)$ with edges indicating braid relations, and they calculate the diameter of this graph. Building on Reiner's proof [14] that the expected number of braid moves in a random element of $R(w_0)$ equals one, Schilling, Thiéry, White, and Williams [17] extend this probabilistic result to the case of one particular commutation class of reduced words for the longest element.

Although braid classes and commutation classes are highly related, recognition of this fact has thus far been under-utilized in the literature. In this paper, we seek to remedy that by studying the relationship between the number of reduced words of a permutation and the numbers of commutation classes and braid classes that it has. Our driving principle is to utilize the fact that commutation classes and braid classes are partitions of the same set. This allows us to leverage one against the other when studying them in tandem. Theorem 3.6 illustrates this approach by providing upper and lower bounds on $|R(w)|$ in terms of the number of braid and commutation classes. More precisely,

$$|B(w)| + |C(w)| - 1 \leq |R(w)| \leq |B(w)| \cdot |C(w)|,$$

where we define this notation in the next section. We demonstrate that these bounds are sharp by characterizing in Proposition 3.8 (and Corollary 3.10) and Proposition 3.15 those permutations that achieve the upper and lower bounds, respectively. Those collections of permutations are enumerated in Corollaries 3.11 and 3.16. In Conjecture 4.2, we suggest an alternate approach to studying these relationships in terms of intervals in the weak order, rather than working directly with the Coxeter relations on reduced words.

We focus on reduced words for elements of the symmetric group, and Section 2 provides the required background on Coxeter groups, as well as the relevant terminology and notation used throughout the paper. In addition, we recall facts about the graph with vertex set $R(w)$ and edges indicating a single commutation or braid move, and we discuss initial results about the quantity and structure of braid classes. This sets the stage for Section 3, which contains the majority of our results, including those discussed above. The paper concludes in Section 4 with a proposed connection to aspects of intervals in the weak order.