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Diversity of uniform intersecting families ☆

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ABSTRACT

A family $\mathcal{F} \subset 2^{[n]}$ is called *intersecting*, if any two of its sets intersect. Given an intersecting family, its *diversity* is the number of sets not passing through a fixed most popular element of the ground set. Peter Frankl made the following conjecture: for $n > 3k > 0$ any intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ has diversity at most $\binom{n-3}{k-2}$. This is tight for the following "two out of three" family: $\{F \in \binom{[n]}{k} : |F \cap [3]| \geq 2\}$. In this note we prove this conjecture for $n \geq ck$, where c is a constant independent of n and$k$. In the last section, we discuss the case $2k < n < 3k$ and show that one natural generalization of Frankl's conjecture does not hold.

1. Introduction

We denote $[n] := \{1, \ldots, n\}$, $2^{[n]} := \{S : S \subset [n]\}$ and $\binom{[n]}{k} := \{S : S \subset [n], |S| = k\}$. Any subset of $2^{[n]}$ we call a *family*. A family $\mathcal{F} \subset 2^{[n]}$ is called *intersecting*, if any two of its sets intersect. The *degree* δ_i of an element $i \in [n]$ is the number of sets from $\mathcal{F}$ containing i. We denote by $\Delta(\mathcal{F})$ the largest degree of an element: the maximum of δ_i over $i \in [n]$. The *diversity* $\gamma(\mathcal{F})$ of $\mathcal{F}$ is the number of sets, not containing the element of the largest degree: $\gamma(\mathcal{F}) := |\mathcal{F}| - \Delta(\mathcal{F})$.

The study of intersecting families started from the famous Erdős–Ko–Rado theorem [5], and since then much effort was put into understanding the structure of large intersecting families. The EKR theorem states that the largest uniform intersecting family consists of all sets containing a given element, that is, the maximal family of diversity 0. The Hilton–Milner theorem [9] gives the largest size of the family with diversity at least 1. Frankl's theorem [6], especially in its strengthened version due to Kupavskii and Zakharov [17] bounds the size of the families with diversity at least $\binom{n-u-1}{n-k-1}$, where $3 \leq u \leq k$ is a fixed real number. We also refer to [16], where, among other results, a conclusive version of this theorem was obtained.

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Theorem 1 (*[17]*). *Let $n > 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Then, if $\gamma(\mathcal{F}) \ge \binom{n-u-1}{n-k-1}$ for some real $3 \le u \le k$, then*

$$|\mathcal{F}| \le \binom{n-1}{k-1} + \binom{n-u-1}{n-k-1} - \binom{n-u-1}{k-1}. \tag{1}$$

It is easy to see that the theorem above is sharp for each integer $u \in [3, k]$: consider the families

$$\mathcal{A}_u := \{F \in \binom{[n]}{k} : F \supset [2, u+1] \text{ or } 1 \in F, F \cap [2, u+1] \neq \emptyset\}, \qquad u \in [2, k].$$

The family $\mathcal{A}_3$ has diversity $\binom{n-4}{k-3}$ and size $\binom{n-1}{k-1} + \binom{n-4}{k-3} - \binom{n-4}{k-1} = 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$. The family $\mathcal{A}_2$ has the same size as $\mathcal{A}_3$ (and this is why the case $u = 2$ does not appear in Theorem 1), but the diversity of $\mathcal{A}_2$ is bigger: it is equal to $\binom{n-3}{k-2}$.

The following problem was suggested by Katona and addressed by Lemons and Palmer [18]: what is the maximum diversity of an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$? They found out that for $n > 6k^3$ we have $\gamma(\mathcal{F}) \le \binom{n-3}{k-2}$, with the equality possible only for $\mathcal{A}_2$ and some of its subfamilies. Recently, Frankl [7] (Theorem 2.4) proved that $\gamma(\mathcal{F}) \le \binom{n-3}{k-2}$ for all $n \ge 6k^2$, and conjectured that the same holds for $n > 3k$.

The purpose of this note is to prove the following theorem

Theorem 2. *There exists a constant C, such that for any $n > Ck > 0$ any intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\gamma(\mathcal{F}) \le \binom{n-3}{k-2}$. Moreover, if $\gamma(\mathcal{F}) = \binom{n-3}{k-2}$, then $\mathcal{F}$ is a subfamily of an isomorphic copy of $\mathcal{A}_2$.*

We note that a somewhat similar proof strategy, which first uses results on Boolean functions to obtain some rough structure for the problem, and then uses combinatorics to obtain a precise result, was recently used by Keller and Lifshitz [14] in a much more general setting.

2. Proof of Theorem 2

The following theorem, proven by Dinur and Friedgut [3], is the main ingredient in the proof. We say that a family $\mathcal{J} \subset 2^{[n]}$ is a *j-junta*, if there exists a subset $J \subset [n]$ of size j (the *center* of the junta), such that the membership of a set in $\mathcal{F}$ is determined only by its intersection with J, that is, for some family $\mathcal{J}^* \subset 2^J$ (the *defining family*) we have $\mathcal{F} = \{F : F \cap J \in \mathcal{J}^*\}$.

Theorem 3 (*[3]*). *For any integer $r \ge 2$,there exist functions $j(r), c(r)$, such that for any integers $1 < j(r) < k < n/2$, if $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family with $|\mathcal{F}| \ge c(r)\binom{n-r}{k-r}$, then there exists an intersecting j-junta $\mathcal{J}$ with $j \le j(r)$ and*

$$|\mathcal{F} \setminus \mathcal{J}| \le c(r)\binom{n-r}{k-r}. \tag{2}$$

We start the proof of the theorem. Choose C sufficiently large (its choice will become clear later), $n > Ck > 0$ and an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$. Then, applying Theorem 3 with $r = 5$, we get that there exists a j-junta $\mathcal{J}, j \le j(5)$, such that $|\mathcal{F} \setminus \mathcal{J}| \le c(5)\binom{n-5}{k-5} < \binom{n-5}{k-4}$, where the second inequality holds provided C is large enough.

The first step is to show that, unless $\mathcal{J} = \mathcal{A}_2$, we have $\gamma(\mathcal{F}) < \binom{n-3}{k-2}$.

Proposition 4. *Consider an intersecting j-junta $\mathcal{J} \subset 2^{[n]}$, with center $J \subset [n]$, $|J| = j$, and defined by an intersecting family $\mathcal{J}^* \subset 2^J$. Then $\mathcal{J}$ satisfies one of the two following properties:*

- *$\mathcal{J}$ is contained in a family isomorphic to $\mathcal{A}_2$.*
- *There exists $i \in J$, such that all sets from $\mathcal{J}^*$ of size at most 2 contain i.*

Proof. Note that the intersecting families of $\le$ 2-element sets which cannot be pierced by a single element are isomorphic to $\binom{[3]}{2}$. Therefore, the junta that does not fall into the second category must