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Pattern-avoiding polytopes

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ABSTRACT

Two well-known polytopes whose vertices are indexed by permutations in the symmetric group $\mathfrak{S}_n$ are the permutohedron P_n and the Birkhoff polytope B_n. We consider polytopes $P_n(\Pi)$ and $B_n(\Pi)$, whose vertices correspond to the permutations in $\mathfrak{S}_n$ avoiding a set of patterns Π. For various choices of Π, we explore the Ehrhart polynomials and h^*-vectors of these polytopes as well as other aspects of their combinatorial structure.

For $P_n(\Pi)$, we consider all subsets $\Pi \subseteq \mathfrak{S}_3$ and are able to provide results in most cases. To illustrate, $P_n(123, 132)$ is a Pitman–Stanley polytope, the number of interior lattice points in $P_n(132, 312)$ is a derangement number, and the normalized volume of $P_n(123, 231, 312)$ is the number of trees on n vertices.

The polytopes $B_n(\Pi)$ seem much more difficult to analyze, so we focus on four particular choices of Π. First we show that the $B_n(231, 321)$ is exactly the Chan–Robbins–Yuen polytope. Next we prove that for any Π containing $\{123, 312\}$ we have $h^*(B_n(\Pi)) = 1$. Finally, we study $B_n(132, 312)$ and $\widetilde{B}_n(123)$, where the tilde indicates that we choose vertices corresponding to alternating permutations avoiding the pattern 123. In both cases we use order complexes of posets and techniques from toric algebra to construct regular, unimodular triangulations of the polytopes. The posets involved turn out to be isomorphic to the lattices of Young diagrams contained in a certain shape, and this permits us to give an exact expression for the normalized volumes of the corresponding polytopes via the hook formula. Finally, Stanley's theory of (P, ω)-partitions allows us to show that their h^*-vectors are symmetric and unimodal.

Various questions and conjectures are presented throughout.

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1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group on $1, 2, \ldots, n$ and $\mathfrak{S} = \cup_{n\geq 0}\mathfrak{S}_n$. Let $\pi \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_n$. We say that σ *contains the pattern* π if there is some substring σ' of σ whose elements have the same relative order as those in π. Alternatively, we view σ' as *standardizing* to π by replacing the smallest element of σ' with 1, the next smallest by 2, and so on. If there is no such substring then we say that σ *avoids the pattern* π. If $\Pi \subseteq \mathfrak{S}$, then we say σ *avoids* Π if σ avoids every element of Π. We will use the notation

$$\mathrm{Av}_n(\Pi) := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \Pi\}.$$

Note this is *not* the avoidance class of Π which is the union of these sets over all n.

A polytope $P \subseteq \mathbb{R}^n$ is the convex hull of finitely many points, written $P = \mathrm{conv}\{v_1, \ldots, v_k\}$. Equivalently, a polytope may be described as a bounded intersection of finitely many half-spaces. The *dimension* of P is the dimension of its affine span. We think of vectors in $\mathbb{R}^n$ as columns and use $a^T b$ to denote the usual inner product of $a, b \in \mathbb{R}^n$. An affine hyperplane H determined by the equation $a^T x = b$ for some $a, b \in \mathbb{R}^n$ is called *supporting* if $a^T p \geq b$ for every $p \in P$. Some texts, such as [19], insist that $H \cap P$ be nonempty; our definition aligns with those found in [6,37]. If H is a supporting hyperplane, then the set $H \cap P$ is called a *face* of P and is a subpolytope of P. Faces of dimension 0 are *vertices*, faces of dimension 1 are called *edges*, and faces of dimension $\dim P - 1$ are called *facets*. Additionally, we say a polytope is a *lattice polytope* if each vertex is an element of $\mathbb{Z}^n$. Lattice polytopes have long found connections with permutations, in particular via the permutohedron and Birkhoff polytope.

The *permutohedron* is defined as

$$P_n := \mathrm{conv}\{(a_1, \ldots, a_n) \mid a_1 \cdots a_n \in \mathfrak{S}_n\}.$$

We will often make no distinction between a permutation and its corresponding point in $\mathbb{R}^n$. This polytope was first described in [30] and has connections to the geometry of flag varieties as well as representations of GL_n. We refer to [42] for general background regarding permutohedra.

The *Birkhoff polytope* is the polytope

$$B_n := \mathrm{conv}\left\{X = (x_{i,j}) \in (\mathbb{R}_{\geq 0})^{n\times n} \mid \sum_{i=1}^{n} x_{i,j} = \sum_{j=1}^{n} x_{i,j} = 1 \text{ for all } i, j\right\}.$$

The Birkhoff–von Neumann Theorem states that the vertices of B_n are the permutation matrices.

In this article, we describe a natural blending of pattern avoidance with the permutohedron and the Birkhoff polytope. Specifically, for any set of patterns Π, we define $P_n(\Pi)$ to be the subpolytope of P_n obtained by taking the convex hull of those vertices corresponding to permutations in $\mathrm{Av}_n(\Pi)$. The polytope $B_n(\Pi)$ is defined similarly. We study the Ehrhart polynomials and h^*-vectors of these polytopes as well as other aspects of their combinatorial structure.

The rest of this paper is organized as follows. In Section 2 we review some basic notions about pattern avoidance and polytopes which will be needed throughout. Section 3 focuses on the permutohedron case $P_n(\Pi)$. We first show in Proposition 3.2 that the action of a certain subgroup of the dihedral group of the square produces unimodularly equivalent polytopes. We then consider all possible $\Pi \subseteq \mathfrak{S}_3$ and are able to provide results for most of the orbits of this action. Specific propositions are listed in Table 1. As a sampling, $P_n(123, 132)$ is a Pitman–Stanley polytope, the number of interior lattice points in $P_n(132, 312)$ is a derangement number, and the normalized volume of $P_n(123, 231, 312)$ is the number of trees on n vertices.

The Π-avoiding Birkhoff polytope appears to be much harder to analyze in general. So we concentrate on four specific examples. In Section 4, we show that $B_n(231, 321)$ is a polytope studied by Chan, Robbins, and Yuen. Next we prove that for any Π containing the permutations 123 and 312 we have $h^*(B_n(\Pi)) = 1$. In Section 5 we begin our study of $B_n(132, 312)$ and $\widetilde{B}_n(123)$, the tilde indicating that we choose vertices corresponding to alternating permutations avoiding the pattern 123. In both cases we use order complexes of posets and techniques from toric algebra to construct regular, unimodular triangulations of the polytopes. The posets involved turn out to be isomorphic