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Pentagons in triangle-free graphs

Bernard Lidický [a], Florian Pfender [b]

[a] *Department of Mathematics, Iowa State University, Ames, IA, United States*
[b] *Department of Mathematical and Statistical Sciences, University of Colorado Denver, Denver, CO, United States*

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ABSTRACT

For all $n \geq 9$, we show that the only triangle-free graphs on n vertices maximizing the number 5-cycles are balanced blow-ups of a 5-cycle. This completely resolves a conjecture by Erdős, and extends results by Grzesik and Hatami, Hladký, Král', Norin and Razborov, where they independently showed this same result for large n and for all n divisible by 5.
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1. Introduction

In 1984, Erdős [1] conjectured that for all $n \geq 5$, the balanced blow-up of a 5-cycle maximizes the number of 5-cycles in the class of triangle-free graphs on n vertices. *Balanced blow-up* means in this context that every vertex of the 5-cycle is replaced by an independent set of size $\lfloor \frac{n}{5} \rfloor$ or $\lceil \frac{n}{5} \rceil$, and edges are replaced by complete bipartite graphs between the sets. This conjecture, in a way, supports the Meta-Theorem saying that among all triangle-free graphs, this blow-up of C_5 is the "least" bipartite graph.

In 2012, Grzesik [2], and independently in 2013, Hatami, Hladký, Král', Norin and Razborov [3] settled this conjecture asymptotically by showing that any n-vertex triangle-free graph has at most $\frac{5!}{5^5}\binom{n}{5}(1+o(1)) = 0.0384\binom{n}{5}(1+o(1))$ 5-cycles. Furthermore, the second group of authors showed uniqueness of the extremal graphs for most values of n.

Theorem 1 (*Hatami et al. [3]*)**.** *Let n be either divisible by* 5*, or large enough. The maximum number of copies of a* 5*-cycle in triangle-free graphs on n vertices is*

$$\prod_{i=0}^{4} \left\lfloor \frac{n+i}{5} \right\rfloor .$$

E-mail addresses: lidicky@iastate.edu (B. Lidický), florian.pfender@ucdenver.edu (F. Pfender).

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Moreover, the only triangle-free graphs on n vertices maximizing the number 5*-cycles are balanced blow-ups of a 5-cycle.*

Both papers [2] and [3] use the theory of flag algebras, a theory developed by Razborov [7], which can find inequalities of subgraph densities in graph limits with the help of semi-definite programming.

In 2011, Michael [6] observed that the Möbius ladder on 8 vertices (i.e. the 8-cycle with all diagonals added) contains the same number of 5-cycles as the balanced blow-up of C_5. In this note we resolve the remaining cases of the conjecture. We also employ flag algebra techniques to find helpful inequalities, and then we use stability results to settle the conjecture for $n \geq 10$. Finally, enumeration is used to deal with $n \leq 9$. Note that our analysis to show uniqueness is significantly simpler than the analysis in [3].

Theorem 2. *For all n, the maximum number of copies of a* 5*-cycle in any triangle-free graph on n vertices is*

$$\prod_{i=0}^{4} \left\lfloor \frac{n+i}{5} \right\rfloor .$$

Moreover, the only triangle-free graphs on $n \geq 5$ *vertices maximizing the number* 5*-cycles are balanced blow-ups of a* 5*-cycle, and the Möbius ladder* ML_8 *for the special case of* $n = 8$*.*

For the ease of presentation, we will use a simplified language of graph limits. No understanding past the following definitions is required to follow this note. A *graphon B of a graph G* with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ is a symmetric function $B : [0, 1]^2 \to \{0, 1\}$ with

$$B(x, y) = \begin{cases} 1, & \text{if } \dfrac{i-1}{n} \leq x < \dfrac{i}{n}, \dfrac{j-1}{n} \leq y < \dfrac{j}{n}, v_i v_j \in E(G), \\ 0, & \text{if } \dfrac{i-1}{n} \leq x < \dfrac{i}{n}, \dfrac{j-1}{n} \leq y < \dfrac{j}{n}, v_i v_j \notin E(G). \end{cases}$$

For a set $X = \{x_1, x_2, \dots, x_k\}$ of real numbers in [0, 1], $B(X)$ is the graph with vertex set X and edge set $\{\{x_i, x_j\} : B(x_i, x_j) = 1\}$. For a graph H, the *induced density* of H in B is

$$d_B(H) = \mathbb{P}(B(X) \cong H),$$

where X is an $|H|$-set chosen uniformly at random from [0, 1]. Note that this quantity equals the limit of the densities of H in balanced blow-ups of G on N vertices, where $N \to \infty$. This notion of a graphon of a graph is a subclass of the much more general objects called graphons in the theory of dense graph limits developed by Lovász and Szegedy ([5], see the book by Lovász [4] for a thorough introduction to the theory). We use a different parametrization to follow the theory of flag algebras by Razborov.

In the following, we assume some familiarity with flag algebras, and we provide only a very brief reminder of the definitions. For a proper introduction to flag algebras, see [7].

We will need the notion of a flag. A *flag* is a graph F on n vertices, in which some vertices, say the first k vertices $S = \{x_1, \dots, x_k\} \subseteq V(F)$, are labeled with distinct labels. The *type* σ of F is the labeled graph induced on S. Two flags are isomorphic if there exists an isomorphism of the underlying graphs which induces an isomorphism of the labeled vertices. The graphon B of the flag F is the graphon of the unlabeled graph underlying F, together with the set $Y = \{\frac{0}{n}, \frac{1}{n}, \dots, \frac{k-1}{n}\} \subset [0, 1]$. Note that $B(Y) \cong \sigma$. For any flag H of type σ, we then define the induced density similarly as for graphs as

$$d_B(H) = \mathbb{P}(B(X \cup Y) \cong H),$$

where X is an $(|H| - |Y|)$-set chosen uniformly at random from [0, 1].

A *flag algebra* $\mathcal{F}^\sigma$ consists of formal linear combinations of flags of the type σ, with the canonical definitions for addition and scalar multiplication. The definition of the density function is extended using linearity. A multiplication of flags is also defined in a way that it naturally corresponds to the multiplication of flag densities in graphons of flags.

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