European Journal of Combinatorics 73 (2018) 211–230

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Dioid partitions of groups

Ishay Haviv, Dan Levy

The School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 2 Rabenu Yeruham St., Tel-Aviv 61083, Israel

A R T I C L E I N F O

Article history:
Received 18 August 2017
Accepted 25 June 2018

A B S T R A C T

A partition of a group is *a dioid partition* if the following three conditions are met: The setwise product of any two parts is a union of parts, there is a part that multiplies as an identity element, and the inverse of a part is a part. This kind of a group partition was first introduced by Tamaschke in 1968. We show that a dioid partition defines a dioid structure over the group, analogously to the way a Schur ring over a group is defined. After proving fundamental properties of dioid partitions, we focus on three part dioid partitions of cyclic groups of prime order. We provide classification results for their isomorphism types as well as for the partitions themselves.

1. Introduction

A *dioid* is a triple $(D, \oplus, \otimes)$, where D is a set and $\oplus$ and $\otimes$ are two binary operations over D, such that $(D, \oplus, \otimes)$ is a semiring and D is canonically ordered by $\oplus$ (see Definition 2.1). Dioids and rings share most of their axioms, except that in the ring case, the additive substructure is a commutative group (hence cannot be canonically ordered by $\oplus$ — see Remark 2.3). In addition to the inherent algebraic interest in a structure which is both similar to and clearly distinct from a ring, dioids have attracted much attention over the last 30 years due to their interesting practical applications. These include the solution of a variety of optimal path problems in graph theory, extensions of classical algorithms for solving shortest paths problems with time constraints, data analysis techniques, hierarchical clustering and preference analysis, algebraic modeling of fuzziness and uncertainty, and discrete event systems in automation (see [1,3,4,8,9]).

As is well known, groups give rise in a natural way to rings via the construction of group rings and its generalization to Schur rings. The aim of the present paper is to point out a similar connection between groups and dioids. We consider the following definition, first introduced by Tamaschke [20] (see Remark 1.8).

E-mail address: danlevy@mta.ac.il (D. Levy).

https://doi.org/10.1016/j.ejc.2018.06.004
0195-6698/

Definition 1.1 (*d-partition for short*). A *dioid partition* of a group G is a partition Π of G which satisfies the following:

a. The closure property: $\forall \pi_1, \pi_2 \in \Pi$, the setwise product $\pi_1\pi_2$ is a union of parts of Π.
b. Existence of an identity: There exists $e \in \Pi$ satisfying $e\pi = \pi e = \pi$ for every $\pi \in \Pi$. We will also write e_Π.
c. The inverse property: Π is invariant under the $G \to G$ mapping defined by $g \longmapsto g^{-1}$ for all $g \in G$.

When $e = \{1_G\}$ we shall say that Π is a 1d-partition.

Note that certain classical group partitions satisfy the above conditions and thus form d-partitions.

Example 1.2. 1. If A is a subgroup of the group G then the set $\Pi = \{AxA | x \in G\}$ of all double cosets of A in G is a d-partition of G with identity A and $(AxA)^{-1} = Ax^{-1}A$. The case $A = \{1_G\}$ yields the *singleton partition* $\Pi = \{\{g\} | g \in G\}$ of G.

2. Let Π be the set of all conjugacy classes of the group G. Here the identity element is $Cl(1_G) = \{1_G\}$, and $Cl(x)^{-1} = Cl\left(x^{-1}\right)$.

The following theorem shows that a d-partition of a group G gives rise to a dioid.

Theorem 1.3. *Let G be a group and Π a d-partition of G. Let D_Π be the set of all possible unions of parts of Π. Denote set union by $\oplus$ and setwise product of subsets of G by $\otimes$. Then $(D_\Pi, \oplus, \otimes)$ is a dioid.*

A dioid which is constructed from a d-partition Π of G as in Theorem 1.3, will be called a *Schur dioid* over G (induced by Π). The reason for this terminology is the strong resemblance of Schur dioids to Schur rings (see Section 3.2). In fact, every Schur ring defines a Schur dioid in a natural way (see Proposition 3.14). The study of Schur rings (and the more general structures of association schemes and coherent configurations) is a well-developed research area which goes back to the work of Issai Schur [19]. Its first systematic treatment was carried by H. Wielandt [24], and since then it has been further developed and found fruitful applications (see [15,16]). Our interest in studying Schur dioids was motivated by an observation in [5].[1]

In the first part of the paper we consider general properties of d-partitions. We prove several basic results and discuss the analogies and direct relations between Schur dioids and Schur rings. We then present several constructions of d-partitions from other d-partitions. The following theorem provides conditions for refining d-partitions and for making them coarser, and shows that every d-partition can be refined to a 1d-partition. For part (c) see Remark 4.1.

Theorem 1.4. *Let G be a group. For a d-partition Π of G and $A \le G$ set*

$$\Pi_{<A} := \{\pi \in \Pi | \pi A = A\pi = A\},$$
$$\Pi_{>A} := \{\pi \in \Pi | \pi A = A\pi = \pi\} \setminus \{A\}.$$

a. *Let Π be a d-partition of G, let $A \le G$, and let $\Pi' := \Pi_{>A} \cup \{A\}$. Then Π' is a d-partition of G with $e_{\Pi'} = A$ if and only if $\{\Pi_{<A}, \Pi_{>A}\}$ is a partition of Π.*
b. *Let $A \le G$, let Π_A be a d-partition of A, and let Π' be a d-partition of G with $e_{\Pi'} = A$. Then $\Pi := \Pi_A \cup \left(\Pi' \setminus \{A\}\right)$ is a d-partition of G with $e_\Pi = e_{\Pi_A}$. In particular, for any d-partition Π' of G with $e_{\Pi'} = A$ there exists a 1d-partition Π of G such that $\Pi'_{>A} \subseteq \Pi$.*
c. *Let Π be a d-partition of G, and let $A \le G$ be such that $\Pi_{<A}$ is a partition of A. Then $\Pi_{<A}$ is a d-partition of A with $e_{\Pi_{<A}} = e_\Pi$ and*

$$\Pi' := \{A\pi A | \pi \in \Pi \setminus \Pi_{<A}\} \cup \{A\}$$

is a d-partition of G with $e_{\Pi'} = A$.

[1] In [5] it is proved that any group with a BN-pair and a finite Weyl group is the product of three conjugates of the Borel subgroup B [5, Theorem 1.5]. The calculations needed for the proof take place in the Schur dioid induced by the double cosets of B.