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Triangle-free induced subgraphs of the unitary polarity graph

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ABSTRACT

Let $\perp$ be a unitary polarity of a finite projective plane π of order q^2. The unitary polarity graph is the graph with vertex set the points of π where two vertices x and y are adjacent if $x \in y^\perp$. We show that a triangle-free induced subgraph of the unitary polarity graph of an arbitrary projective plane has at most $(q^4+q)/2$ vertices. When π is the Desarguesian projective plane $\mathrm{PG}(2, q^2)$ and q is even, we show that the upper bound is asymptotically sharp, by providing an example on $q^4/2$ vertices. Finally, the case when π is the Figueroa plane is discussed.

1. Introduction

Let π be a finite projective plane of order q. A polarity $\perp$ of π is an involutory bijective map sending points to lines and lines to points which preserves incidence. A point x of π is said to be absolute if $x \in x^\perp$. The **polarity graph** $\mathcal{G}(\pi, \perp)$ is the graph with vertex set the points of π where two vertices x and y are adjacent if $x \in y^\perp$. Remark that we could have defined the graph $\mathcal{G}(\pi, \perp)$ equivalently with the lines of π as vertices. This graph is not simple: every absolute point gives rise to a loop. A classical theorem by Baer [1] states that every polarity has at least $q+1$ absolute points, which implies that there a polarity graph has at least $q+1$ loops. With a slight abuse of notation we will identify the vertices of the polarity graph with the points (or lines) of the plane. We will say for example that a point x is adjacent to another point y. For all definitions and notions regarding projective planes and polarities not mentioned in Section 2, we refer the reader to [2,21,22].

Polarity graphs and their properties have been the subject of study over the last few years. Questions regarding their independence number [15,26,27], chromatic number [32] and other properties

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have been posed and (partially) answered. The motivation behind this line of research lies first of all in the fact that these graphs possess a lot of structure and interesting features. More importantly, polarity graphs are related to some classes of problems in extremal graph theory, among which Ramsey problems and Turán-type problems. For example in the latter, Füredi [13,14] has shown that the unique graph with the most edges among all graphs on q^2+q+1 vertices not containing C_4 as a subgraph is the polarity graph, where $\perp$ is an orthogonal polarity, i.e., a polarity with $q+1$ absolute points (which is the least possible as we already mentioned).

Recently, Loucks and Timmons [24] have drawn attention to the following problem.

Question 1.1. What is the largest set of non-absolute vertices in $\mathcal{G}(\pi, \perp)$ inducing a triangle-free subgraph?

Note that only non-absolute vertices are considered, since triangles in a polarity graph cannot contain absolute vertices, see also Section 2.

This problem first appeared in [27] in the context of extremal graph theory, where the authors considered the case when $\pi = \mathrm{PG}(2, q)$ and $\perp$ an orthogonal polarity. They used a construction due to Parsons [28] to obtain an upper bound on the independence number of a 3-uniform hypergraph which first appeared in [23]. Parsons' construction on which they relied is exactly a triangle-free induced subgraph of $\mathcal{G}(\pi, \perp)$.

Loucks and Timmons also mention that one of the motivations behind this question is from Turán-type problems. In particular, we are interested in the maximum number of edges in an n-vertex graph without C_3 or C_4 as a subgraph. Indeed, it is natural to approach this problem by considering C_4-free graphs with many edges, and finding a C_3-free subgraph thereof.

In this article, we investigate the case when $\perp$ is a unitary polarity. Then the order of the projective plane is necessarily a square, say q^2, there are q^3+1 absolute points and the set of absolute points forms a unital $\mathcal{U}$. Note that there are unitals which do not arise from a unitary polarity, see [2] for further results on this topic. We denote by $\mathrm{UP}(q^2)$ a unitary polarity graph for an arbitrary projective plane of order q^2. In the first part of the paper, by refining the techniques used in [24], we obtain the following upper bound for a triangle-free induced subgraph of $\mathrm{UP}(q^2)$.

Theorem 1.2. *Let S be a subset of non-absolute vertices of* $\mathrm{UP}(q^2)$ *inducing a triangle-free subgraph, then*

$$|S| \leq \frac{q^4+q}{2}.$$

Moreover, if equality holds and ℓ *is a line of* π*, then* $|\ell \cap S| \in \{\frac{q^2-q}{2}, \frac{q^2+q}{2}\}$.

In the second part of the paper we deal with the case when π is the Desarguesian projective plane $\mathrm{PG}(2, q^2)$. We will denote this graph by $\mathrm{DUP}(q^2)$. When q is even, we are able to show that the upper bound is asymptotically sharp.

Theorem 1.3. *For q even, there exists a set of non-absolute vertices of* $\mathrm{DUP}(q^2)$ *inducing a triangle-free subgraph of size* $q^4/2$.

In the last part of the paper we consider the case when π is the Figueroa plane $\mathcal{F}$. The plane $\mathcal{F}$ is obtained by the Desarguesian plane $\mathrm{PG}(2, q^3)$, by distorting certain lines. It is known that every polarity of $\mathcal{F}$ induces a polarity of $\mathrm{PG}(2, q^3)$ [19, Theorem 4.2]. Vice versa, under certain assumptions, a polarity $\perp$ of the Desarguesian projective plane $\mathrm{PG}(2, q^3)$ gives rise to a polarity $\perp'$ of the Figueroa plane $\mathcal{F}$. In this case, we show that a triangle-free induced subgraph of $\mathcal{G}(\mathrm{PG}(2, q^3), \perp)$ gives rise to a triangle-free induced subgraph of $\mathcal{G}(\mathcal{F}, \perp')$. This answers a question of Loucks and Timmons [24, Question 1.4] in the case when π is the Figueroa plane of order q^6.

2. Preliminaries about UP(q^2)

Before we can prove these results, we need some structural information about $\mathrm{UP}(q^2)$, in particular about the neighbourhood structure, see [22, Chapter XII]. If x is a point of π, then $x^\perp$ denotes its *polar*