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Random 4-regular graphs have 3-star decompositions asymptotically almost surely

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ABSTRACT

Barát and Thomassen conjectured in 2006 that the edges of every planar 4-regular 4-edge-connected graph can be decomposed into copies of the star with 3 leaves. Shortly afterward, Lai constructed a counterexample to this conjecture. Using the small subgraph conditioning method of Robinson and Wormald, we prove that a random 4-regular graph has an S_3-decomposition asymptotically almost surely, provided the number of vertices is divisible by 3.

1. Introduction

A question that has garnered much study is whether the edges of a graph G can be decomposed into copies of a small fixed subgraph, say F. Of course, some natural divisibility conditions arise for such a decomposition, namely that $e(F)$ must divide $e(G)$. Kotzig observed [9] that if G is connected and $e(G)$ is even, then G decomposes into copies of S_2, the star with 2 leaves. What happens for larger F; in particular, are there natural conditions when F is isomorphic to the S_3, the star with 3 leaves? Not much was known about this problem until Thomassen's breakthrough results [14] on the weak 3-flow conjecture. In particular, we note the following theorem which follows from a more general theorem of Lovász, Thomassen, Wu, and Zhang [11].

Theorem 1.1. *If $F \simeq S_k$, the star with k leaves, and G is a d-edge-connected graph such that k divides $e(G)$ and $2 \leq k \leq \lceil d/2 \rceil$, then the edge set of G decomposes into copies of F.*

In fact, Theorem 1.1 is tight for $k \geq 3$. To see this, first note that if $k > d$, then K_k is a d-edge-connected graph with no S_k decomposition. For $k \leq d$ with $k \geq 3$ and $k > \lceil d/2 \rceil$, consider k copies of

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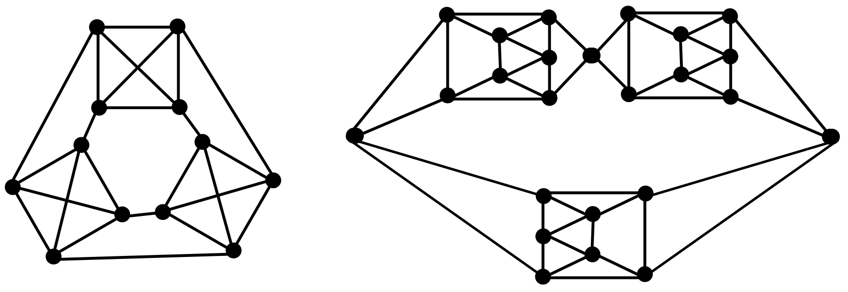

Fig. 1. On the left is a non-planar 4-regular 4-edge-connected graph with no S_3-decomposition. On the right is Lai's planar construction.

K_d with edges added so that the resulting graph G is d-regular and d-edge-connected. If there existed an S_k-*decomposition* of G (a decomposition of the edges of G into copies of S_k), then because $k > d/2$, such a decomposition would naturally partition the vertices into $\frac{d}{2k}v(G) = \frac{d^2}{2}$ centers of the stars and $\frac{2k-d}{2k}v(G) = \frac{d(2k-d)}{2}$ non-centers. However, the non-centers must form an independent set, and thus, there are at most k of them, the desired contradiction (because $k < \frac{d(2k-d)}{2}$ when $2k - d \geq 2$).

Thus, when F is isomorphic to S_3, Theorem 1.1 implies that a d-regular d-edge-connected graph G has an F-decomposition if $d \geq 5$ and 3 divides $e(G)$. For $d = 3$, it is easy to observe that a 3-regular graph has an S_3-decomposition if and only if it is bipartite. As for the case when $d = 4$, the construction in Fig. 1 on the left provides a non-planar example of a 4-regular 4-edge-connected graph G where 3 divides $e(G)$ but G does not have an S_3-decomposition. This led Barát and Thomassen [2], who knew of this example, to conjecture in 2006 that every planar 4-regular 4-edge-connected graph G where 3 divides $e(G)$ has an S_3-decomposition. Unfortunately in the following year, Lai presented an infinite family of clever counterexamples (replicated in Fig. 1 on the right) to this nice conjecture [10].

Given that a typical d-regular graph is d-edge-connected, a natural setting in which to study these questions is that of random regular graphs. We utilize the *configuration model* (also known as the *pairing model*) introduced by Bollobás [4]. Let $d \geq 1$ and dn be even; we take a total of dn points and partition them into n *cells* each consisting of exactly d points. Any perfect matching of $\frac{dn}{2}$ pairs of points is said to be a *configuration*, also known as a *pairing*. Each configuration corresponds to a multigraph (possibly with loops) where the cells are vertices and the pairs are edges. We denote the uniform probability space of configurations by $\mathcal{P}_{n,d}$. In the configuration model, we choose an element of $\mathcal{P}_{n,d}$ uniformly at random and discard the result if the corresponding d-regular multigraph has loops or parallel edges. This was shown to be equivalent to choosing a d-regular (simple) graph on n vertices uniformly at random (c.f. Wormald's survey paper [17] for more details).

Observe that in any simple 4-regular graph G, an orientation of the edges of G in which every in-degree is either 4 or 1 (alternatively every out-degree is either 0 or 3) is equivalent to an S_3-*decomposition*, that is a decomposition of the edges of G into copies of S_3; namely, the vertices with out-degree 3 are the centers of the stars formed by their out-edges. In light of this, we consider orientations of the edges of a configuration where the out-degree of every cell is 0 or 3, where the *out-degree* of a cell is defined to be the number of points in the cell that are the tail of some edge in the orientation. We call such an orientation a $(3, 0)$-*orientation*.

The main result of this paper is as follows. Note that all asymptotics in this article are as n tends to infinity along positive integers divisible by 3.

Theorem 1.2. *A configuration in* $\mathcal{P}_{n,4}$ *has a* $(3, 0)$-*orientation asymptotically almost surely, provided that* n *is divisible by* 3.

Any 4-regular (simple) graph G on n vertices corresponds to exactly $(4!)^n = 24^n$ configurations in $\mathcal{P}_{n,4}$. Because each such graph corresponds to the same number of configurations, it follows that G is a (uniformly) random 4-regular (simple) graph in the configuration model. The probability that a