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On the irregularity of uniform hypergraphs[☆]

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ABSTRACT

Let H be an r-uniform hypergraph on n vertices and m edges, and let d_i be the degree of $i \in V(H)$. Denote by $\varepsilon(H)$ the difference between the spectral radius of H and the average degree of H. Also, denote

$$s(H) = \sum_{i\in V(H)} \left| d_i - \frac{rm}{n} \right|, \quad v(H) = \frac{1}{n} \sum_{i\in V(H)} d_i^{\frac{r}{r-1}} - \left(\frac{rm}{n}\right)^{\frac{r}{r-1}}.$$

In this paper, we investigate the irregularity of r-uniform hypergraph H with respect to $\varepsilon(H)$, $s(H)$ and $v(H)$, which extend relevant results to uniform hypergraphs.

1. Introduction

Let $G = (V(G), E(G))$ be an undirected graph with n vertices and m edges without loops and multiple edges, where $V(G) = [n] := \{1, 2, \dots, n\}$. A graph G is regular if all its vertices have the same degree; otherwise it is irregular. In many applications and problems it is of importance to know how irregular a given graph is. Various measures of graph irregularity have been proposed and studied, see, for example, [2,4,10,15,16] and references therein.

We first survey some known parameters used as measures of irregularity as well as their respective properties. In 1957, Collatz and Sinogowitz [4] showed that the spectral radius $\rho(G)$ of a graph G is greater than or equal to the average degree $\overline{d}(G)$, and equality holds if and only if G is regular. This fact allows us to consider the difference $\varepsilon(G) = \rho(G) - \overline{d}(G)$ as a relevant measure of irregularity of G. The authors also proved that, for $n \leq 5$, the maximum value of $\varepsilon(G)$ is $\sqrt{n-1} - 2 + 2/n$ and the maximum is attained for the star S_n. Fifty years later, Aouchiche et al. [1] conjectured that the most irregular connected graph on n $(n \geq 10)$ vertices is a pineapple graph, which is obtained by appending

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some pendant edges to a vertex of a complete graph. Recently, this conjecture was proved by Tait and Tobin [26]. In 1992, Bell [2] suggested making the variance $v(G)$ of the vertex degrees of G as a measure of the irregularity, i.e.,

$$v(G) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{2m}{n}\right)^2.$$

The author compared $\varepsilon(G)$ and $v(G)$ for various classes of graphs, and showed that they are not always compatible. Also, the most irregular graphs according to these measures were determined for certain classes of graphs. In 2006, Nikiforov [15] introduced

$$s(G) = \sum_{i\in V(G)} \left| d_i - \frac{2m}{n} \right|$$

as a new measure of the irregularity of a graph G, and showed several inequalities with respect to $\varepsilon(G)$, $s(G)$ and $v(G)$ as follows:

$$\frac{v(G)}{2\sqrt{2m}} \le \rho(G) - \frac{2m}{n} \le \sqrt{s(G)}. \tag{1.1}$$

In particular, for a bipartite graph G with m edges and partition $V(G) = V_1 \dot\cup V_2$, Nikiforov [15] defined

$$s_2(G) = \sum_{i\in V_1} \left| d_i - \frac{m}{n_1} \right| + \sum_{i\in V_2} \left| d_i - \frac{m}{n_2} \right|$$

as a more relevant irregularity parameter than $s(G)$, where $n_1 = |V_1|$, $n_2 = |V_2|$. Also, it was proved that

$$\rho(G) - \frac{m}{\sqrt{n_1 n_2}} \le \sqrt{\frac{s_2(G)}{2}}. \tag{1.2}$$

These irregularity measures as well as other attempts to measure the irregularity of a graph were studied in several works [6–8,10,23].

Our work in the present paper is to study the irregularity of uniform hypergraphs. Denote by $\mathcal{H}(n, m)$ the set of all the r-uniform hypergraphs with n vertices and m edges. Let $H \in \mathcal{H}(n, m)$ be an r-uniform hypergraph, and $\rho(H)$ be the spectral radius of H. In 2012, Cooper and Dutle [5] showed that $\rho(H) \ge rm/n$. It is clear that the equality holds if and only if H is regular by [21, Theorem 2]. Therefore, the value

$$\varepsilon(H) = \rho(H) - \frac{rm}{n}$$

can be viewed as a relevant measure of irregularity of H. Denote

$$s(H) = \sum_{i\in V(H)} \left| d_i - \frac{rm}{n} \right|,$$

where d_i is the degree of vertex i of H. Obviously, $s(H) \ge 0$, with equality if and only if H is regular. Analogous to the graph case, if $H \in \mathcal{H}(n, m)$ is an r-partite r-uniform hypergraph with partition $V(H) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_r$ and $|V_i| = n_i$, $i \in [r]$, we denote

$$s_r(H) = \sum_{i\in [r]} \sum_{j\in V_i} \left| d_j - \frac{m}{n_i} \right|.$$

For an r-uniform hypergraph $H \in \mathcal{H}(n, m)$, we also denote

$$v(H) = \frac{1}{n}\sum_{i=1}^{n} d_i^{\frac{r}{r-1}} - \left(\frac{rm}{n}\right)^{\frac{r}{r-1}}.$$

It follows from Power Mean inequality that $v(H) \ge 0$, and equality holds if and only if H is regular.