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A note on diameter-Ramsey sets

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ABSTRACT

A finite set $A \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ and a finite set $B \subset \mathbb{R}^n$ with $\text{diam}(A) = \text{diam}(B)$ such that whenever B is coloured with r colours, there is a monochromatic set $A' \subset B$ which is congruent to A. We prove that sets of diameter 1 with circumradius larger than $1/\sqrt{2}$ are not diameter-Ramsey. In particular, we obtain that triangles with an angle larger than 135° are not diameter-Ramsey, improving a result of Frankl, Pach, Reiher and Rödl. Furthermore, we deduce that there are simplices which are almost regular but not diameter-Ramsey.

1. Introduction

In this note, we discuss questions related to *Euclidean Ramsey theory*, a field introduced in [1] by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus. A finite set $A \subset \mathbb{R}^d$ is called *Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that in every colouring of $\mathbb{R}^n$ with r colours, there is a monochromatic set $A' \subset \mathbb{R}^n$ which is congruent to A. The problem of classifying which sets are Ramsey has been widely studied and is still open (see [3] for more details).

The *diameter* of a set $P \subset \mathbb{R}^d$ is defined by $\text{diam}(P) := \sup\{\|x - y\| : x, y \in P\}$, where $\|\cdot\|$ denotes the Euclidean norm. Recently, Frankl, Pach, Reiher and Rödl [2] introduced the following stronger property.

Definition 1.1. A finite set $A \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ and a finite set $B \subset \mathbb{R}^n$ with $\text{diam}(A) = \text{diam}(B)$ such that whenever B is coloured with r colours, there is a monochromatic set $A' \subset B$ which is congruent to A.

It follows from the definition that every diameter-Ramsey set is Ramsey. A set $A \subset \mathbb{R}^d$ is called spherical, if it lies on some d-dimensional sphere and the *circumradius* of A, denoted by $\text{cr}(A)$, is the radius of the smallest sphere containing A. (Note that if A is spherical and is not contained in a proper

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subspace of $\mathbb{R}^d$, then there is a unique sphere that contains it.) In [1] it was proved that every Ramsey set must be spherical. Our main result states that every diameter-Ramsey set must also have a small circumradius.

Theorem 1.2. *If $A \subset \mathbb{R}^d$ is a finite, spherical set with circumradius strictly larger than* $\mathrm{diam}(A)/\sqrt{2}$*, then A is not diameter-Ramsey.*

Frankl, Pach, Reiher and Rödl [2, Theorems 3 and 4] proved that acute and right-angled triangles are diameter-Ramsey, while triangles having an angle larger than 150° are not. Theorem 1.2 implies the following improvement.

Corollary 1.3. *Triangles with an angle larger than 135° are not diameter-Ramsey.*

Let us call a d-simplex $A = \{p_1, \dots, p_{d+1}\}$ *ε-almost regular* if

$$\frac{1}{\binom{d+1}{2}} \sum_{1 \le i < j \le d+1} \mathrm{diam}(A)^2 - \left\| p_i - p_j \right\|^2 \le \varepsilon \cdot \mathrm{diam}(A)^2.$$

In [2, Theorem 6, Lemma 4.9] it was further proved that ε-almost regular simplices are diameter-Ramsey for every $\varepsilon \le 1/\binom{d+1}{2}$. This is a rather small class of simplices since $1/\binom{d+1}{2}$ tends to zero, but another corollary of Theorem 1.2 shows that one cannot hope for much more.

Corollary 1.4. *For every $d \in \mathbb{N}$ and every $\varepsilon > \sqrt{d}/\binom{d+1}{2}$, there is an ε-almost regular d-simplex which is not diameter-Ramsey.*

For $d \in \mathbb{N}$ and $r \ge 0$, we denote the closed d-dimensional ball of radius r centred at the origin by $B_d(r)$. We will deduce Theorem 1.2 from the following result.

Theorem 1.5. *For every finite, spherical set $A \subset \mathbb{R}^d$ and every positive number $r <$ $\mathrm{cr}(A)$, there is some $k = k(A, r) \in \mathbb{N}$ such that the following holds. For every $D \in \mathbb{N}$, there is a colouring of $B_D(r)$ with k colours and with no monochromatic, congruent copy of A.*

A result of Matoušek and Rödl [5] shows that the conclusion of Theorem 1.5 does not hold whenever $r > \mathrm{cr}(A)$. We do not know what happens when $r = \mathrm{cr}(A)$.

Remark 1.6. After completing this work, we have learnt that Theorem 1.2 has independently been proved by Frankl, Pach, Reiher and Rödl, with a similar proof (János Pach, private communication).

2. Proofs

2.1. Proof of Theorem 1.5

Fix some finite, spherical $A \subset \mathbb{R}^d$ and some positive number $r < \mathrm{cr}(A)$. The following claim is the key step of the proof.

Claim 2.1. *There exists a constant $c = c(A, r) > 0$ such that for every $D \in \mathbb{N}$ and for every congruent copy A' of A in $B_D(r)$ we have* $\max_{x,y \in A'} (\|x\| - \|y\|) \ge c$.

Proof. First observe that it is sufficient to prove the claim for $D = d + 1$. For $D < d + 1$, this follows immediately from $B_D(r) \subset B_{d+1}(r)$, and for $D > d+1$ we can consider the at most $(d+1)$-dimensional subspace spanned by the vertices of A' and the origin.

Let $E = \{e : A \to B_D(r)\} \subset B_D(r)^{|A|}$ be the set of all embeddings of A to B_D. It is easy to see that, if $e_1, e_2, \dots \in E$ and the pointwise limit $e := \lim_n e_n$ exists, then $e \in E$. Therefore, E is a closed subset of a compact metric space and hence E is compact as well. Define $f : E \to \mathbb{R}$ by

$$f(e) := \max_{x,y \in e(A)} (\|x\| - \|y\|)\,.$$