European Journal of Combinatorics 71 (2018) 55–72

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Shuffles of trees

Eric Hoffbeck [a], Ieke Moerdijk [b]

[a] Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, 99 avenue Jean-Baptiste Clément, F-93430, Villetaneuse, France

[b] Department of Mathematics, Utrecht University, PO BOX 80.010, 3508 TA Utrecht, The Netherlands

ARTICLE INFO

Article history:
Received 27 June 2017
Accepted 15 February 2018
Available online 10 March 2018

ABSTRACT

We discuss a notion of shuffle for trees which extends the usual notion of a shuffle for two natural numbers. We give several equivalent descriptions, and prove some algebraic and combinatorial properties. In addition, we characterize shuffles in terms of open sets in a topological space associated to a pair of trees. Our notion of shuffle is motivated by the theory of operads and occurs in the theory of dendroidal sets, but our presentation is independent and entirely self-contained.

0. Introduction

For two natural numbers p and q the set of (p, q)-shuffles plays a central rôle in many parts of algebra, topology, probability theory and combinatorics. For example, they occur in the description of the coalgebra and Hopf algebra structures on exterior or tensor algebras [14], and in the description of the Eilenberg–Zilber map for the homology of a product of two topological spaces [10]. The name shuffle refers back to the fact that the (p, q)-shuffles are shuffles of linear orders of length p and q (rather than just sets of cardinality p and q), like shuffling two decks of p and q cards respectively, as studied in [1].

The goal of this paper is to study a notion of shuffle of two trees, rather than just of linear orders. Several such notions already occur in the literature, for example in the context of automata theory and formal languages [8]. Our notion is different from these. It specialises to the standard one if the two trees happen to be linear orders, and seems very natural from the point of view of non-deterministic programming semantics where the trees describe programs. Such shuffles of trees also enter in the description of a free resolution of the Boardman–Vogt tensor product of operads [3], and related to this, play a crucial rôle in the homotopy theory of dendroidal sets [4].

E-mail addresses: hoffbeck@math.univ-paris13.fr (E. Hoffbeck), i.moerdijk@uu.nl (I. Moerdijk).

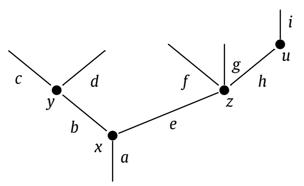
https://doi.org/10.1016/j.ejc.2018.02.035

In this paper, we will present a purely combinatorial and self-contained discussion of this notion of shuffle of two trees, which is motivated by, but can be read completely independently from the theory of operads and dendroidal sets. In particular, we shall consider questions concerning shuffles which have not been addressed in that context, such as: What is the structure of the set of shuffles of two trees? How is the number of shuffles related to the size of the trees? etc. That second question has a very simple answer in terms of binomial coefficients in the linear case, but seems quite intractable for general trees, as we will explain.

The plan of the paper, then, is as follows. In a first section, we will give what we believe is the most accessible definition of a shuffle of two trees, and illustrate it by various examples. In a subsequent section, we prove that this definition is equivalent to several others, the most concise one being that a shuffle of two trees S and T is a maximal subtree of the product partial order $S \times T$ containing all the pairs of leaves (cf. Proposition 2.3). In a third section, we discuss some aspects of the number of shuffles of two trees. It is an open question whether one can find a comprehensible closed formula expressing the number of shuffles of two trees S and T in terms of the size (height, width, etc.) of S and T. We present some upper and lower bounds, and show that in the case where T is a linear tree of length n and S is fixed, this number is a polynomial in n with rational coefficients, of which the degree and leading coefficient can be described quite simply in terms of the size of S. In a fourth section, we will show that the set of shuffles of two trees S and T carries the natural structure of a distributive lattice, which is rather rigid in the sense that in most cases it has no automorphisms other than the ones coming from automorphisms of S or T. This description also leads to the observation that shuffles can be composed. Indeed, two shuffles, one between S and T, and the other between R and S, naturally give rise to a third shuffle between R and T. More technically, we prove that trees and shuffles between them form a category enriched in distributive lattices. In a final section, mainly added for motivation and background, we very briefly discuss in which way shuffles of trees naturally occur in topology, in operad theory and in the theory of dendroidal sets.

1. Definition and first examples

In this section we present a first definition of a shuffle of two trees. What we mean by a "tree" in this context is a finite connected graph without cycles, whose external edges are open, i.e., connected to one vertex only. One of these external edges is selected as the root, and the other external edges are called the leaves of the tree. The chosen root provides an orientation, pictured downwards towards the root. Each vertex will have one outgoing edge (towards the root), and a strictly positive number of incoming edges to which we will refer as the valence of the vertex (see also Remark 2.8). Here is a picture of the type of tree that we shall consider.



This is a tree with five leaves, and root edge a. There are four vertices, of valence two, two, three and one. The vertex which has the root as output edge will be referred to the root vertex. For an arbitrary tree T, we will write $E(T)$ and $V(T)$ for its sets of edges and vertices, respectively. Moreover, we will denote by r_T its root edge (and later in Section 4 its root vertex).

Remark 1.1. When drawing a tree, the picture automatically provides the tree with a planar structure. We do not presuppose our trees to have any planar structure however, and a picture like the one above but with the leaves c and d interchanged, for example, represents the same tree.

Remark 1.2. There are various ways to think about such a tree which are relevant for what follows. When reading the tree from top to bottom, we think of the edges as *objects* and of the vertices as