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Möbius coinvariants and bipartite edge-rooted forests

Woong Kook, Kang-Ju Lee

Department of Mathematical Sciences, Seoul National University, Seoul, Republic of Korea

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ABSTRACT

The Möbius coinvariant $\mu^{\perp}(G)$ of a graph G is defined to be the Möbius invariant of the dual of the cycle matroid of G. This invariant is known to equal the rank of the reduced homology of the cycle matroid complex of G. For a complete graph K_{m+1}, W. Kook gave an interpretation of $\mu^{\perp}(K_{m+1})$ as the number of edge-rooted forests in K_m. In this paper, we obtain a new combinatorial interpretation of $\mu^{\perp}(K_{m+1,n+1})$ as the number of B-edge-rooted forests in $K_{m,n}$, which is a bipartite analogue of the previous result.

Based on these interpretations, we will give new bijective proofs of the formulas for $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$ given by I. Novik, A. Postnikov, and B. Sturmfels in terms of the Hermite polynomials. In addition, we will construct a homology basis for the cycle matroid complex of $K_{m+1,n+1}$ indexed by the B-edge-rooted forests. Also we will discuss the Möbius coinvariant of *bi-coned* graphs which generalize complete bipartite graphs.

1. Introduction

The Möbius invariant $\mu(M)$ of a matroid M is defined to be $|\mu_{L(M)}(\hat{0}, \hat{1})|$ where $\mu_{L(M)}$ is the Möbius function on the lattice of flats $L(M)$ of M. We define the Möbius coinvariant of M to be $\mu^{\perp}(M) := \mu(M^*)$ where M^* is the dual matroid of M. It is well-known that $\mu^{\perp}(M)$ equals an evaluation $T_M(0, 1)$ of the Tutte polynomial $T_M(x, y)$ of M, which also equals the rank of the reduced homology of the independent set complex $IN(M)$ of M [2,3]. For a graph G, we define its Möbius coinvariant to be $\mu^{\perp}(G) := \mu(M(G)^*)$ where $M(G)$ is the cycle matroid of G. In this paper, we will give a new interpretation of the Möbius coinvariant for a class of graphs generalizing complete bipartite graphs.

Some of the previous results that motivated the current work are as follows. In the context of hyperplane arrangement and commutative algebra, D. Bayer, S. Popescu, and B. Sturmfels posed

E-mail addresses: woongkook@snu.ac.kr (W. Kook), leekj0706@snu.ac.kr (K.-J. Lee).

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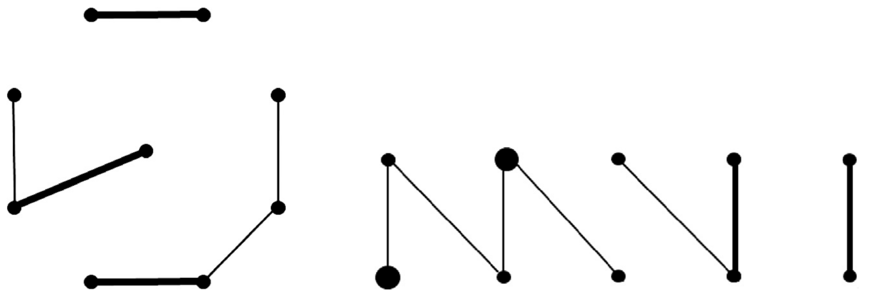

(a) An edge-rooted forest. (b) A B-edge-rooted forest.

Fig. 1. An edge-rooted forest and a B-edge-rooted forest.

a problem of computing $\mu^{\perp}(G)$ [1]. In response to this problem, I. Novik, A. Postnikov, and B. Sturmfels [14] gave formulas for $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$ for complete graphs K_{m+1} and complete bipartite graphs $K_{m+1,n+1}$, studying cographic ideals and hyperplane arrangements. They expressed these formulas in terms of the Hermite polynomials, a generating function for partial matchings. In a study of group action on the homology of matroid complexes, W. Kook [6] or [11, Theorem 20] gave a simple combinatorial interpretation for $\mu^{\perp}(K_{m+1})$, and more generally for the Möbius coinvariant of a *coned graph* [8], via *edge-rooted forests* (see Fig. 1(a)). However, a direct correspondence relating the edge-rooted forests and the formula for $\mu^{\perp}(K_{m+1})$ has not appeared before. We will establish this correspondence in this paper.

More importantly, we will give a new combinatorial interpretation for the Möbius coinvariant of a complete bipartite graph $K_{m+1,n+1}$, which can be outlined as follows. A tree in $K_{m,n}$ is *bi-rooted* if it has two root vertices, one in each bipartite set of $K_{m,n}$. A tree in $K_{m,n}$ is edge-rooted if exactly one edge is marked as an *edge-root*. Define a *B-edge-rooted forest* in $K_{m,n}$ to be a spanning forest with exactly one component bi-rooted and the remaining components edge-rooted (see Fig. 1(b)). By identifying all spanning trees with zero internal activity, we will show that $\mu^{\perp}(K_{m+1,n+1})$ equals the number of all B-edge-rooted forests in $K_{m,n}$ (see Theorem 3.4), which is an analogue of the result for a complete graph.

As a consequence of our combinatorial interpretations for the invariants $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$, we will give new bijective proofs of their formulas given by I. Novik, A. Postnikov, and B. Sturmfels [14]. In Theorem 4.3, we will show that $\mu^{\perp}(K_{m+1})$ equals

$$\sum_{k\geq 1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (2k \cdot m^{m-1-2k})(2k-1)!!$$

where the kth term in this sum is the number of edge-rooted forests in K_m with k components. In Theorem 4.8, we will show that $\mu^{\perp}(K_{m+1,n+1})$ equals

$$\sum_{k=0}^{\min(m-1,n-1)} \binom{m}{k}\binom{n}{k} k!(m-k)(n-k)n^{m-1-k}m^{n-1-k}$$

where the kth term in the sum is the number of B-edge-rooted forests in $K_{m,n}$ with $k+1$ components (*i.e.*, the number of B-edge-rooted forests with one bi-rooted tree and k edge-rooted trees). As a tool for proving this fact, we develop a bipartite analogue of Lemma 4.2 about the number of the vertex-rooted forests in a complete graph with given roots (see Theorem 4.6).

B-edge-rooted forests are not only a combinatorial interpretation, but also a "code" for constructing a homology basis for the cycle matroid complex of $K_{m+1,n+1}$. W. Kook [6] showed how edge-rooted forests in K_m could be used to construct a homology basis for the independent set complex $I(K_{m+1}) := IN(M(K_{m+1}))$ (see Example 3 in [11]). This basis also reveals that the action of the