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Binomial edge ideals of bipartite graphs

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ABSTRACT

Binomial edge ideals are a noteworthy class of binomial ideals that can be associated with graphs, generalizing the ideals of 2-minors. For bipartite graphs we prove the converse of Hartshorne's Connectedness Theorem, according to which if an ideal is Cohen–Macaulay, then its dual graph is connected. This allows us to classify Cohen–Macaulay binomial edge ideals of bipartite graphs, giving an explicit and recursive construction in graph-theoretical terms. This result represents a binomial analogue of the celebrated characterization of (monomial) edge ideals of bipartite graphs due to Herzog and Hibi (2005).

1. Introduction

Binomial edge ideals were introduced independently in [10] and [17]. They are a natural generalization of the ideals of 2-minors of a $(2 \times n)$-generic matrix [3]: their generators are those 2-minors whose column indices correspond to the edges of a graph. In this perspective, the ideals of 2-minors are binomial edge ideals of complete graphs. On the other hand, binomial edge ideals arise naturally in Algebraic Statistics, in the context of conditional independence ideals, see [10, Section 4].

More precisely, given a finite simple graph G on the vertex set $[n] = \{1, \dots, n\}$, the *binomial edge ideal* associated with G is the ideal

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \text{ is an edge of } G) \subset R = K[x_i, y_i : i \in [n]].$$

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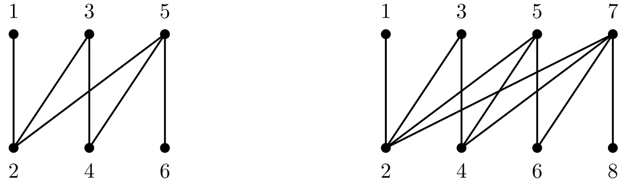

(a) The graph F_3. (b) The graph F_4.

Fig. 1. The basic blocks.

Binomial edge ideals have been extensively studied, see e.g. [1,5,6,13–15,18,19]. Yet a number of interesting questions is still unanswered. In particular, many authors have studied classes of Cohen–Macaulay binomial edge ideals in terms of the associated graph, see e.g. [1,5,13,18,19]. Some of these results concern a class of chordal graphs, the so-called *closed graphs*, introduced in [10], and their generalizations, such as block and generalized block graphs [13].

In the context of squarefree monomial ideals, any graph can be associated with the so-called *edge ideal*, whose generators are monomials of degree 2 corresponding to the edges of the graph. Herzog and Hibi, in [9, Theorem 3.4], classified Cohen–Macaulay edge ideals of bipartite graphs in purely combinatorial terms. In the same spirit, we provide a combinatorial classification of Cohen–Macaulay binomial edge ideals of bipartite graphs.

To this aim we exploit the dual graph of an ideal. Following the notation used in [2], we recall this notion and some facts.

Let I be an ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of I. The *dual graph* $\mathcal{D}(I)$ is a graph with vertex set $[r]$ and edge set

$$\{\{i, j\} : \mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_j) - 1 = \mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{p}_j) = \mathrm{ht}(I)\}.$$

This notion was originally studied by Hartshorne in [8] in terms of *connectedness in codimension one*. By [8, Corollary 2.4], if A/I is Cohen–Macaulay, then the algebraic variety defined by I is connected in codimension one, hence I is unmixed by [8, Remark 2.4.1]. The connectedness of the dual graph translates in combinatorial terms the notion of connectedness in codimension one, see [8, Proposition 1.1]. Thus, if A/I is Cohen–Macaulay, then $\mathcal{D}(I)$ is connected. The converse does not hold in general, see for instance Remark 5.1. We will show that for binomial edge ideals of connected bipartite graphs this is indeed an equivalence. In geometric terms, this means that the algebraic variety defined by J_G is Cohen–Macaulay if and only if it is connected in codimension one.

We now describe the explicit structure of the bipartite graphs in the classification. For the terminology about graphs we refer to [4]. First we present a family of bipartite graphs F_m whose binomial edge ideal is Cohen–Macaulay, and we prove that, if G is connected and bipartite, then J_G is Cohen–Macaulay if and only if G can be obtained recursively by gluing a finite number of graphs of the form F_m via two operations.

Basic blocks: For every $m \geq 1$, let F_m be the graph (see Fig. 1) on the vertex set $[2m]$ and with edge set

$$E(F_m) = \{\{2i, 2j - 1\} : i = 1, \dots, m, j = i, \dots, m\}.$$

Notice that F_1 is the single edge $\{1, 2\}$ and F_2 is the path of length 3.

Operation $*$: For $i = 1, 2$, let G_i be a graph with at least one vertex f_i of degree one, i.e., a *leaf* of G_i. We denote the graph G obtained by identifying f_1 and f_2 by $G = (G_1, f_1) * (G_2, f_2)$, see Fig. 2(a). This is a particular case of an operation studied by Rauf and Rinaldo in [18, Section 2].

Operation $\circ$: For $i = 1, 2$, let G_i be a graph with at least one leaf f_i, v_i its neighbour and assume $\deg_{G_i}(v_i) \geq 3$. We define $G = (G_1, f_1) \circ (G_2, f_2)$ to be the graph obtained from G_1 and G_2 by removing the leaves f_1, f_2 and identifying v_1 and v_2, see Fig. 2(b).

For both operations, if it is not important to specify the vertices f_i or it is clear from the context, we simply write $G_1 * G_2$ or $G_1 \circ G_2$.