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Asymptotics of the number of standard Young tableaux of skew shape

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ABSTRACT

We give new bounds and asymptotic estimates on the number of standard Young tableaux of skew shape in a variety of special cases. Our approach is based on Naruse's hook-length formula. We also compare our bounds with the existing bounds on the numbers of linear extensions of the corresponding posets.

1. Introduction

The classical *hook-length formula* (HLF) allows one to compute the number $f^\lambda = |\mathrm{SYT}(\lambda)|$ of standard Young tableaux of a given shape [16]. This formula had profound applications in Enumerative and Algebraic Combinatorics, Discrete Probability, Representation Theory and other fields (see e.g. [41,42,49]). Specifically, the HLF allows to derive asymptotics for f^λ for various families of "large" partitions λ. This was famously used to compute the diagram of a random representation of S_n with respect to the Plancherel measure $(f^\lambda)^2/n!$, see [29,51] (see also [3,47]).

For skew shapes λ/μ, little is known about the asymptotics, since there is no multiplicative formula for $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$. For large $n = |\lambda/\mu|$, the asymptotics are known for a few special families of skew shapes (see [1]), and for fixed μ (see [35,40,46]). In this paper we show that the "naive HLF" gives good approximations for $f^{\lambda/\mu}$ in many special cases of interest.

Formally, let λ be a partition of n. Denote by $f^\lambda = |\mathrm{SYT}(\lambda)|$ the number of standard Young tableaux of shape λ. We have:

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}, \tag{HLF}$$

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where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the square $u = (i, j)$. Now, let λ/μ be a skew shape, $n = |\lambda/\mu|$. By analogy with the HLF, define

$$F(\lambda/\mu) := \frac{n!}{\prod_{u\in\lambda/\mu} h(u)}, \qquad \text{(naive HLF)}$$

where $h(u)$ are the usual hook-lengths in λ.[1]

Our main technical tool is Theorem 3.3, which gives

$$(*) \qquad F(\lambda/\mu) \le f^{\lambda/\mu} \le \xi(\lambda/\mu)F(\lambda/\mu),$$

where $\xi(\lambda/\mu)$ is defined in Section 3. These bounds turn out to give surprisingly sharp estimates for $f^{\lambda/\mu}$, compared to standard bounds on the number $e(\mathcal{P})$ of linear extensions for general posets.[2] We also give several examples when the lower bound is sharp but not the upper bound, and vice versa (see e.g. Sections 9.1 and 10.2).

Let us emphasize an important special case of *thick ribbons* δ_{k+r}/δ_k, where $\delta_k = (k-1, k-2, \ldots, 2, 1)$ denotes the *staircase shape*. The following result illustrates the strength of our bounds (cf. Section 3.3).

Theorem 1.1. *Let $\upsilon_k = (\delta_{2k}/\delta_k)$, where $\delta_k = (k-1, k-2, \ldots, 2, 1)$. Then*

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \le \frac{1}{n}\left(\log f^{\upsilon_k} - \frac{1}{2} n\log n\right) \le \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\upsilon_k| = k(3k-1)/2$.

Here the LHS ≈ -0.3237, and the RHS ≈ -0.0621. Note that the numbers $f^{\delta_{k+r}/\delta_k}$ of standard Young tableaux for thick ribbons (see [44, A278289]) have been previously considered in [2], but the tools in that paper apply only for $r \to \infty$.

We should mention that in the theorem and in many other special cases, the leading terms of the asymptotics are easy to find. Thus most of our effort is over the lower order terms which are much harder to determine (see Section 12). In fact, it is the lower order terms that are useful for applications (see Sections 8.3 and 12.6).

The rest of the paper is structured as follows. We start with general results on linear extensions (Section 2), standard Young tableaux of skew shape (Section 3), and excited diagrams (Section 4). We then proceed to our main results concerning the asymptotics for $f^{\lambda/\mu}$ in the following cases:

(1) when both λ, μ have the *Thoma–Vershik–Kerov limit* (Section 5). Here the Frobenius coordinates scale linearly and $f^{\lambda/\mu}$ grow exponentially.
(2) when both λ, μ have the *stable shape limit* (Section 6). Here the row and column lengths scale as $\sqrt{n}$, and $f^{\lambda/\mu} \approx \sqrt{n!}$ up to an exponential factor.
(3) when λ/μ have *subpolynomial depth* (Section 7). Here both the row and column lengths of λ/μ grow as $n^{o(1)}$, and $f^{\lambda/\mu} \approx n!$ up to a factor of *intermediate growth* (i.e. super-exponential and subfactorial), which can be determined by the depth growth function.
(4) when λ/μ is a large ribbon hook (Section 9). Here λ/μ scale linearly along a fixed curve, and $f^{\lambda/\mu} \approx n!$ up to an exponential factor.
(5) when λ/μ is a *slim shape* (Section 10). Here μ is fixed and $\ell, \lambda_\ell/\ell \to \infty$, where $\ell = \ell(\lambda)$. Here $f^{\lambda/\mu} \sim f^\lambda f^\mu/|\mu|!$.

We illustrate these cases with various examples. Further examples and more specialized applications are given in Sections 8 and 11. We conclude with final remarks in Section 12.

[1] Note that $F(\lambda/\mu)$ is not necessarily an integer.
[2] Both $e(\mathcal{P})$ and $f^{\lambda/\mu}$ are standard notation in respective areas. For the sake of clarify and to streamline the notation, we use $e(\lambda/\mu) = |\mathrm{SYT}(\lambda/\mu)|$ throughout the paper (except for the Introduction and Final Remarks Sections 1 and 12).