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On the degree of regularity of a certain quadratic Diophantine equation

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ABSTRACT

We show that, for every positive integer r, there exists an integer $b = b(r)$ such that the 4-variable quadratic Diophantine equation $(x_1 - y_1)(x_2 - y_2) = b$ is r-regular. Our proof uses Szemerédi's theorem on arithmetic progressions.

1. Introduction

Denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. Given a polynomial $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, let $D(f)$ denote the corresponding Diophantine equation

$$f(x_1, x_2, \ldots, x_n) = 0.$$

This equation is said to be r-*regular*, for some integer $r \geq 1$, if for every r-coloring of $\mathbb{N}_+$, there is a monochromatic solution to it. It is said to be *regular* if it is r-regular for all $r \geq 1$. The *degree of regularity* of $D(f)$, denoted $\mathrm{dor}(D(f))$, is defined to be infinite if $D(f)$ is regular, or else, it is the largest r such that $D(f)$ is r-regular. Determining the degree of regularity of a given Diophantine equation is difficult in general, even if it is linear.

In this paper, we shall consider the 4-variable Diophantine quadratic equation

$$(x_1 - y_1)(x_2 - y_2) = b, \tag{1}$$

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denoted $Q(b)$, where b is a given positive integer. This equation is not regular. Indeed, it is not b-regular, and actually not even s-regular where $s = \lfloor\sqrt{b}\rfloor + 1$, as witnessed by the s-coloring given by the class mod s; for if x_1, y_1, x_2, y_2 are all congruent mod s, then $(x_1 - y_1)(x_2 - y_2)$ is divisible by s^2, and hence cannot equal b since $s^2 > b$. That is, we have $\mathrm{dor}(Q(b)) \leq \lfloor\sqrt{b}\rfloor$. Our purpose in this paper is to show that, nevertheless, the numbers $\mathrm{dor}(Q(b))$ are unbounded as b varies. Here is our main result.

Theorem 1.1. *Given a positive integer r, there is a positive integer $b = b(r)$ such that the equation $(x_1 - y_1)(x_2 - y_2) = b$ is r-regular.*

A more specific version is stated and proved in Section 3.

Our motivation to study this particular quadratic equation comes from previous work on the linear version

$$(x_1 - y_1) + (x_2 - y_2) = b,$$

and more generally on the $2k$-variable linear Diophantine equation

$$(x_1 - y_1) + \cdots + (x_k - y_k) = b, \tag{2}$$

the object of the following conjecture by Fox and Kleitman [6].

Conjecture 1.2. *Let $L_k(b)$ denote Eq. (2). Then*

$$\max_{b\in\mathbb{N}_+} dor(L_k(b)) = 2k - 1.$$

If true, that estimate would be best possible, since it is shown in [6] that $\mathrm{dor}(L_k(b)) \leq 2k - 1$ for all $k, b \geq 1$. See [1,2] for solutions of the Fox–Kleitman conjecture for $k = 2$ and 3, respectively, and [12] for a very recent full proof.

Note that the solved case $k = 2$ of the conjecture and Theorem 1.1 imply a sharp contrast between the additive and the multiplicative versions of the equation, namely $\max_b \mathrm{dor}(L_2(b)) = 3$ versus $\max_b \mathrm{dor}(Q(b)) = \infty$.

1.1. Contents

Here is a brief description of the contents of the paper. In Section 2, we recall some classical problems on partition regularity. In Section 3, after recalling Szemerédi's theorem on arithmetic progressions, we prove our main result on the unboundedness of $\mathrm{dor}(Q(b))$ as b varies. In Section 4, after setting up specific tools for the task at hand, we provide estimates for one of the numbers $M(k, \delta)$ involved in Szemerédi's theorem. In Section 5, we determine all $b \geq 1$ for which $Q(b)$ is 2-regular, as well as the smallest $b \geq 1$ for which $Q(b)$ is 3-regular. The last section is devoted to a few remarks and open questions.

2. Background

We first recall some background results and problems on partition regularity. The following abridged version of a theorem of Rado [9–11] characterizes the regular linear homogeneous equations on $\mathbb{Z}$.

Theorem 2.1 (*Radó's Theorem, Abridged Version*)**.** *For $n \geq 2$ and $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$, the Diophantine equation*

$$c_1x_1 + \cdots + c_nx_n = 0 \tag{3}$$

is regular if and only $\sum_{i\in I} c_i = 0$ for some non-empty subset $I \subseteq \{1, \ldots, n\}$.