European Journal of Combinatorics 70 (2018) 61–74

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On subgraphs of random Cayley sum graphs

S.V. Konyagin, I.D. Shkredov

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russian Federation

A R T I C L E   I N F O

Article history:
Received 30 November 2016
Accepted 29 November 2017

A B S T R A C T

We prove that asymptotically almost surely, the random Cayley sum graph over a finite abelian group $\mathbf{G}$ has edge density close to the expected one on every induced subgraph of size at least $\log^c|\mathbf{G}|$, for any fixed $c > 1$ and $|\mathbf{G}|$ large enough.

© 2017 Elsevier Ltd. All rights reserved.

1. Introduction

Let A be a subset of an additively written group $\mathbf{G}$. We denote by $\mathbf{Cay}(A, \mathbf{G})$ the *Cayley sum graph* induced by A on $\mathbf{G}$, which is the directed graph on the vertex set $\mathbf{G}$ in which $(x, y) \in \mathbf{G} \times \mathbf{G}$ is an edge if and only if $x + y \in A$ ($x = y$ is allowed). Such graphs are classical combinatorial objects, see, e.g. [2]. Green [3] initiated the study of the *random Cayley sum graph*, considering finite groups $\mathbf{G}$ and selecting A at random by choosing each $x \in \mathbf{G}$ to lie in A independently and at random with probability 1/2. General random graphs are considered in [1]. Results about random Cayley sum graphs can be found, for example, in [3,4,7,6]. R. Mrazović [7] proved the following theorem.

Theorem 1. *Let $\mathbf{G}$ be a finite group and $w : \mathbb{N} \to \mathbb{R}$ be a growing function that tends to infinity. Let $A \subset \mathbf{G}$ be a random subset obtained by putting every element of $\mathbf{G}$ into A independently with probability $\frac{1}{2}$. Then with probability $1 - o(1)$, for all sets $X, Y \subset \mathbf{G}$ with*

$$|X| \geq w(|\mathbf{G}|)\log|\mathbf{G}| \quad \text{and} \quad |Y| \geq w(|\mathbf{G}|)\log^2|\mathbf{G}|$$

one has

$$\sum_{x\in X}\sum_{y\in Y} A(x+y) = \frac{1}{2}|X||Y| + o(|X||Y|), \quad (|\mathbf{G}| \to \infty), \tag{1}$$

where the rate of convergence implied by the o-notation depends only on w.

E-mail addresses: konyagin@mi.ras.ru (S.V. Konyagin), ishkredov@mi.ras.ru (I.D. Shkredov).

https://doi.org/10.1016/j.ejc.2017.11.009
0195-6698/© 2017 Elsevier Ltd. All rights reserved.

In Theorem 1 and throughout, for a subset $A \subseteq \mathbf{G}$ we use the same letter to denote its characteristic function $A : \mathbf{G} \to \{0, 1\}$.

Theorem 1 shows that with high probability, the edge density of the random Cayley sum graph on all induced subgraphs of size at least $\log^{2+\varepsilon}|\mathbf{G}|$, is close to $1/2$.

In the same paper Mrazović [7] showed that there is no C such that the assumption of Theorem 1 can be relaxed to $\min\{|X|, |Y|\} \geq C \log|\mathbf{G}| \log\log|\mathbf{G}|$.

Using some tools from Additive Combinatorics, we show that Theorem 1 can be improved.

Theorem 2. *Let $\mathbf{G}$ be a finite abelian group and $w : \mathbb{N} \to \mathbb{R}$ be a growing function that tends to infinity. Let $A \subset \mathbf{G}$ be a random subset obtained by putting every element of $\mathbf{G}$ into A independently with probability $\frac{1}{2}$. Then with probability $1 - o(1)$, for all sets $X, Y \subset \mathbf{G}$ with*

$$|X| \geq w(|\mathbf{G}|) \log|\mathbf{G}|(\log\log|\mathbf{G}|)^2, \quad |Y| \geq w(|\mathbf{G}|) \log|\mathbf{G}|(\log\log|\mathbf{G}|)^{10},$$

one has

$$\sum_{x\in X} \sum_{y\in Y} A(x+y) = \frac{1}{2}|X||Y| + o(|X||Y|) \quad (|\mathbf{G}| \to \infty),$$

where the rate of convergence implied by the o-notation depends only on w.

Notice that the lower bounds in Theorem 2 can be improved by double-logarithmic factors at most.

Let us say a few words about the proof.

It is shown in [7] that if for some X, Y the sum in the left-hand side of (1) deviates significantly from $\frac{1}{2}|X|\,|Y|$, then the common energy (see the definition in the next section) of X and Y must be close to the trivial upper bound $|X|\,|Y| \min\{|X|, |Y|\}$. Mrazović used a random choice to avoid such a situation (see details in [7]). Using structural results from [9,11], we add one more twist to his argument, proving that X and Y possess large, well–structured subsets. Hence, the number of such pairs of sets is much smaller than the total number of all possible pairs of sets. This is what ultimately allows us to relax the requirement on the sizes of X and Y.

First we consider the case of elementary abelian 2-groups and prove Theorem 2 in this situation (with the lower bound $\min\{|X|, |Y|\} \geq \log^{3/2+\varepsilon}|\mathbf{G}|$) using a variation of the argument from [7]. For such groups the proof is simpler and more transparent. For the general case see Sections 4 and 5.

We thank Rudi Mrazović and Mikhail Gabdullin for the fruitful discussions, and the reviewers for their useful remarks.

2. Definitions and preliminary results

Let $\mathbf{G}$ be an abelian group. The *additive energy* $\mathsf{E}(A, B)$ between subsets $A, B \subseteq \mathbf{G}$ is (see [13])

$$\mathsf{E}(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B \ : \ a_1 + b_1 = a_2 + b_2\}|.$$

By $A \bigsqcup B$ denote the union of two disjoint sets A, B.

Recall a simple lemma, see, e.g., [10, Lemma 12].

Lemma 3. *For any finite sets $X, Y, Z \subset \mathbf{G}$ one has*

$$\mathsf{E}(X \cup Y, Z)^{1/2} \leq \mathsf{E}(X, Z)^{1/2} + \mathsf{E}(Y, Z)^{1/2},$$

and for disjoint union of X and Y the following holds

$$\mathsf{E}(X \sqcup Y, Z) \geq \mathsf{E}(X, Z) + \mathsf{E}(Y, Z).$$

Now let us recall the notions of dissociativity and (additive) *dimension* of a set. A finite set $\Lambda \subset \mathbf{G}$ is called *dissociated* if an equality of the form

$$\sum_{\lambda\in\Lambda} \varepsilon_\lambda \lambda = 0$$

Download English Version:

https://daneshyari.com/en/article/8903596

Download Persian Version:

https://daneshyari.com/article/8903596

Daneshyari.com