European Journal of Combinatorics 70 (2018) 149–154

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

An isoperimetric inequality for antipodal subsets of the discrete cube

David Ellis [a], Imre Leader [b]

[a] School of Mathematical Sciences Queen Mary, University of London, Mile End Road, London, E1 4NS, UK
[b] Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, UK

ARTICLE INFO

Article history:
Received 21 September 2016
Accepted 8 December 2017
Available online 12 January 2018

ABSTRACT

We say a family of subsets of $\{1, 2, \ldots, n\}$ is *antipodal* if it is closed under taking complements. We prove a best-possible isoperimetric inequality for antipodal families of subsets of $\{1, 2, \ldots, n\}$ (of any size). Our inequality implies that for any $k \in \mathbb{N}$, among all such families of size 2^k, a family consisting of the union of two antipodal $(k-1)$-dimensional subcubes has the smallest possible edge boundary.

1. Introduction

Isoperimetric questions are classical objects of study in mathematics. In general, they ask for the minimum possible 'boundary-size' of a set of a given 'size', where the exact meaning of these words varies according to the problem.

The classical isoperimetric problem in the plane asks for the minimum possible perimeter of a shape in the plane with area 1. The answer, that it is best to take a circle, was 'known' to the ancient Greeks, but it was not until the 19th century that this was proved rigorously, by Weierstrass in a series of lectures in the 1870s in Berlin.

The isoperimetric problem has been solved for n-dimensional Euclidean space $\mathbb{E}^n$, for the n-dimensional unit sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$, and for n-dimensional hyperbolic space $\mathbb{H}^n$ (for all n), with the natural notion of boundary in each case, corresponding to surface area for sufficiently 'nice' sets. (For background on isoperimetric problems, we refer the reader to the book of Burago and Zalgaller [2], the surveys of Osserman [6] and of Ros [8], and the references therein.) One of the most well-known open problems in the area is to solve the isoperimetric problem for

E-mail address: d.ellis@qmul.ac.uk (D. Ellis).

https://doi.org/10.1016/j.ejc.2017.12.003
0195-6698/

n-dimensional real projective space $\mathbb{RP}^n$, or equivalently for antipodal subsets of the n-dimensional sphere $\mathbb{S}^n$. (We say a subset $\mathcal{A} \subseteq \mathbb{S}^n$ is *antipodal* if $\mathcal{A} = -\mathcal{A}$.) The conjecture can be stated as follows.

Conjecture 1. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let μ denote the n-dimensional Hausdorff measure on $\mathbb{S}^n$. Let $\mathcal{A} \subseteq \mathbb{S}^n$ be open and antipodal. Then there exists a set $\mathcal{B} \subseteq \mathbb{S}^n$ such that $\mu(\mathcal{B}) = \mu(\mathcal{A})$, $\sigma(\mathcal{B}) \leq \sigma(\mathcal{A})$, and*

$$\mathcal{B} = \{x \in \mathbb{S}^n : \sum_{i=1}^{r} x_i^2 > a\}$$

for some $r \in [n]$ and some $a \in \mathbb{R}$.

Here, if $\mathcal{A} \subseteq \mathbb{S}^n$ is an open set, then $\sigma(\mathcal{A})$ denotes the surface area of $\mathcal{A}$, i.e. the $(n-1)$-dimensional Hausdorff measure of the topological boundary of $\mathcal{A}$.

Only the cases $n = 2$ and $n = 3$ of Conjecture 1 are known, the former being 'folklore' and the latter being due to Ritoré and Ros [7]. In this paper, we prove a discrete analogue of Conjecture 1.

First for some definitions and notation. If X is a set, we write $\mathcal{P}(X)$ for the power-set of X. For $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$, and we let Q_n denote the graph of the n-dimensional discrete cube, i.e. the graph with vertex-set $\mathcal{P}([n])$, where x and y are joined by an edge if $|x \Delta y| = 1$. If $\mathcal{A} \subseteq \mathcal{P}([n])$, we write $\partial\mathcal{A}$ for the *edge-boundary* of $\mathcal{A}$ in the discrete cube Q_n, i.e. $\partial\mathcal{A}$ is the set of edges of Q_n which join a vertex in $\mathcal{A}$ to a vertex outside $\mathcal{A}$. We write $e(\mathcal{A})$ for the number of edges of Q_n which have both end-vertices in $\mathcal{A}$. We say that two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$ are *isomorphic* if there exists an automorphism σ of Q_n such that $\mathcal{B} = \sigma(\mathcal{A})$. Clearly, if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then $|\partial\mathcal{A}| = |\partial\mathcal{B}|$.

The *binary ordering* on $\mathcal{P}([n])$ is defined by $x < y$ iff $\max(x \Delta y) \in y$. An *initial segment of the binary ordering on* $\mathcal{P}([n])$ is the set of the first k (smallest) elements of $\mathcal{P}([n])$ in the binary ordering, for some $k \leq 2^n$. For any $k \leq 2^n$, we write $\mathcal{I}_{n,k}$ for the initial segment of the binary ordering on $\mathcal{P}([n])$ with size k.

Harper [3], Lindsay [5], Bernstein [1] and Hart [4] solved the edge isoperimetric problem for Q_n, showing that among all subsets of $\mathcal{P}([n])$ of given size, initial segments of the binary ordering on $\mathcal{P}([n])$ have the smallest possible edge-boundary.

In this paper, we consider the edge isoperimetric problem for antipodal sets in Q_n. If $x \subseteq [n]$, we define $\overline{x} := [n] \setminus x$, and if $\mathcal{A} \subseteq \mathcal{P}([n])$, we define $\overline{\mathcal{A}} := \{\overline{x} : x \in \mathcal{A}\}$. We say a family $\mathcal{A} \subseteq \mathcal{P}([n])$ is *antipodal* if $\mathcal{A} = \overline{\mathcal{A}}$. This notion is of course the natural analogue in the discrete cube of antipodality in $\mathbb{S}^n$; indeed, identifying $\mathcal{P}([n])$ with $\{-1, 1\}^n \subseteq \sqrt{n} \cdot \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ in the natural way, $x \mapsto \overline{x}$ corresponds to the antipodal map $\mathbf{v} \mapsto -\mathbf{v}$.

We prove the following best-possible edge isoperimetric inequality for antipodal families.

Theorem 2. *Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be antipodal. Then*

$$|\partial\mathcal{A}| \geq |\partial(\mathcal{I}_{n,|\mathcal{A}|/2} \cup \overline{\mathcal{I}_{n,|\mathcal{A}|/2}})|.$$

We remark that Theorem 2 implies that if $\mathcal{A} \subseteq \mathcal{P}([n])$ is antipodal with $|\mathcal{A}| = 2^k$ for some $k \in [n-1]$, then $|\partial\mathcal{A}| \geq |\partial(\mathcal{S}_{k-1} \cup \overline{\mathcal{S}_{k-1}})|$, where $\mathcal{S}_{k-1} := \mathcal{I}_{n,2^{k-1}} = \{x \subseteq [n] : x \subseteq [k-1]\}$ is a $(k-1)$-dimensional subcube. In other words, a union of two antipodal subcubes has the smallest possible edge-boundary, over all antipodal sets of the same size.

To prove Theorem 2, it will be helpful for us to rephrase it slightly. Firstly, observe that for any $\mathcal{A} \subseteq \mathcal{P}([n])$, we have $\partial(\mathcal{A}^c) = \partial\mathcal{A}$, and that for any $k \leq 2^{n-1}$, the family $(\mathcal{I}_{n,k} \cup \overline{\mathcal{I}_{n,k}})^c$ is isomorphic to the family $\mathcal{I}_{n,2^{n-1}-k} \cup \overline{\mathcal{I}_{n,2^{n-1}-k}}$, via the isomorphism $x \mapsto x\Delta\{n\}$. Hence, by taking complements, it suffices to prove Theorem 2 in the case $|\mathcal{A}| \leq 2^{n-1}$.

Secondly, for any family $\mathcal{A} \subseteq \mathcal{P}([n])$, we have

$$2e(\mathcal{A}) + |\partial\mathcal{A}| = n|\mathcal{A}|, \tag{1}$$

so Theorem 2 is equivalent to the statement that if $\mathcal{A} \subseteq \mathcal{P}([n])$ is antipodal, then

$$e(\mathcal{A}) \leq e(\mathcal{I}_{n,|\mathcal{A}|/2} \cup \overline{\mathcal{I}_{n,|\mathcal{A}|/2}}).$$

Note also that if $\mathcal{B}$ is an initial segment of the binary ordering on $\mathcal{P}([n])$ with $|\mathcal{B}| \leq 2^{n-2}$, then $\mathcal{B} \subseteq \{x \subseteq [n] : x \cap \{n-1, n\} = \emptyset\}$ and $\overline{\mathcal{B}} \subseteq \{x \subseteq [n] : \{n-1, n\} \subseteq x\}$, so $\mathcal{B} \cap \overline{\mathcal{B}} = \emptyset$