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Occurrence of right angles in vector spaces over finite fields

Michael Bennett

7208 Overhill Dr. Pomona, NY 10970, United States

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ABSTRACT

Here we examine some Erdős–Falconer-type problems in vector spaces over finite fields involving right angles. Our main goals are to show that

(a) a subset $A \subset \mathbb{F}_q^d$ of size $\geq cq^{\frac{d+2}{3}}$ contains three points which generate a right angle, and

(b) a subset $A \subset \mathbb{F}_q^d$ of size $\geq Cq^{\frac{d+2}{2}}$ contains two points which generate a right angle with the vertex at the origin.

We will also prove that (b) is sharp up to constants and provide some partial results for similar problems related to spread and collinear triples.

1. Introduction

In this paper, we will be looking at some extremal problems in combinatorial geometry similar to those proposed by Erdős and Falconer, but in the setting of finite fields rather than in Euclidean space. Throughout, q will be a power of an odd prime, and we will denote the field with q elements by $\mathbb{F}_q$. We would like to determine how large a subset of $\mathbb{F}_q^d$ (with q much larger than d) needs to be to guarantee that the set contains three points which form a right angle. We will also make some observations about extremal problems regarding "spread". Spread is essentially a finite field analog of the Euclidean angle. When we say that a triple (a, b, c) "generates" an angle (or spread) θ, we mean that the angle between the vectors $a - b$ and $c - b$ is θ. This is, in a way, an extension of the right angle problem, because to say that a triple "generates a right angle" is roughly equivalent to saying that the spread generated by the triple is 1. These notions will be explained much more thoroughly in Section 6. Finally, we will ask how large a subset of $\mathbb{F}_q^d$ needs to be to guarantee that three points in that subset are collinear and provide relatively trivial bounds on this problem, not coming particularly close to a decisive answer to the question.

E-mail address: mike.b.bennett87@gmail.com.

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This paper was motivated by a paper of Harangi et al. which looked at the same problems, as well as many others, in $\mathbb{R}^d$. By looking at the same problems in a different setting, we hope to improve our understanding of the relation between the geometric structure of $\mathbb{R}^d$ and $\mathbb{F}_q^d$.

2. Main theorems

Definition 2.1. We say that an ordered triple of points $(x, y, z) \in \mathbb{F}_q^d \times \mathbb{F}_q^d \times \mathbb{F}_q^d$ forms a right angle if $x, y,$ and z are distinct, and the vectors $x - y$ and $z - y$ have dot product 0.

Theorem 2.2. *If $A, B \subset \mathbb{F}_q^d$, $A \cap B = \emptyset$, and $|A|^2|B| \geq 4q^{d+2}$, then there are $x, y \in A$ and $z \in B$ so that (x, y, z) forms a right angle.*

Following immediately from the theorem, we have the following corollary:

Corollary 2.3. *If $A \subset \mathbb{F}_q^d$ and $|A| \geq 4q^{\frac{d+2}{3}}$ then A contains a right angle.*

That is, given a set A, we may decompose it into disjoint sets A' and B' of roughly equal size and then apply the theorem. We will see that the bound from 2.2 is best possible up to the implied constant. It is not known whether the result is improvable in the form of 2.3.

We will also examine the problem of existence of right angles whose vertex is fixed at the origin:

Theorem 2.4. *If $A \subset \mathbb{F}_q^d$ and $|A| \geq 4q^{\frac{d+2}{2}}$, then there are $x, y \in A$ so that $x \cdot y = 0$. Moreover, if q is a prime, then there exists a subset $B \subset \mathbb{F}_q^d$ so that $|B| = \Omega(q^{\frac{d+2}{2}})$ but $\{x, y \in B : x \cdot y = 0\} = \emptyset$.*

3. Comparison with Euclidean analog

Perhaps the most interesting aspect of these results are how they differ from those found in [7]. In their paper, Harangi et al. solve Euclidean versions of these problems. There they pose the problem in two different ways: For a given angle, how large must a set in $\mathbb{R}^d$ be to guarantee that
(1) that angle is generated by that set?
(2) an angle with size within δ of the given angle is generated by that set?

When we say "how large", we are referring to the Hausdorff dimension of the set. In the paper they discover that 90° angles are a singular case. In the sense of question 1, the critical dimension for 90° angles is between $\frac{d}{2}$ and $\frac{d+1}{2}$. That is, there are examples of sets of Hausdorff dimension arbitrarily close to $\frac{d}{2}$ which contain no 90° angles, while any compact set with Hausdorff dimension larger than $\frac{d+1}{2}$ *must* contain a 90° angle (which they prove rather concisely in the paper). For 0 and 180°, the critical dimension is $d - 1$ (because of the $(d - 1)$-sphere), while for other angles only partial answers exist. Question 2 is completely resolved in this paper, where they find that a set of Hausdorff dimension 1 always has an angle very close to 90°, while a set of lower dimension need not. It is known that any sufficiently large finite set of points contains three points at an angle as close to 180° as desired (and therefore close to 0° as well) (see [6]). Interestingly, $60^\circ$and 120° angles also have a separate result that depends on δ. For any other given angle, however, a set of Hausdorff dimension that increases to ∞ as d goes to ∞ can be constructed so that a neighborhood of that angle is avoided.

In Euclidean space, Hausdorff dimension is a natural way to classify the dimension of an arbitrary set. The best analog to this classification in vector spaces over finite fields is to think of the dimension of a set S as $\approx \log_q(|S|)$. This is why, in the finite field setting, the quantity we are most interested in is the exponent α in $|S| = cq^\alpha$. In extremal geometry, problems that can be solved in Euclidean space can often be solved in a similar manner over finite fields. Consequently, one might expect the exponent in the finite field setting to match the Hausdorff dimension in the Euclidean setting. This is why the case of right angles is interesting here. We find that the critical dimension in vector spaces over finite fields is bounded above by $\frac{d+2}{3}$, which is significantly different from the (question 1) threshold found in [7]. As far as the Euclidean analog of Theorem 2.2, consider the set $(0, \infty) \times \cdots \times (0, \infty) \subset \mathbb{R}^d$. The dot product of any two elements in this set must be positive, and the set clearly has Hausdorff dimension d, so this also presents a discrepancy between the continuous and finite settings.