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Highly connected subgraphs of graphs with given independence number

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ABSTRACT

Let G be a graph on n vertices with independence number α. We prove that if n is sufficiently large ($n \geq \alpha^2 k + 1$ will do), then G always contains a k-connected subgraph on at least n/α vertices. The value of n/α is sharp, since G might be the disjoint union of α equally-sized cliques. For $k \geq 3$ and $\alpha = 2, 3$, we shall prove that the same result holds for $n \geq 4(k-1)$ and $n \geq \frac{27(k-1)}{4}$ respectively, and that these lower bounds on n are sharp.

1. Introduction

When can we find a large highly connected subgraph of a given graph G? A classical theorem due to Mader [10] (see also [6]) states that if G has average degree at least $4k$, then G contains a k-connected subgraph H. Mader's theorem does not give a lower bound on the order of H. If G is dense (for instance if $\delta(G)$, the minimum degree of G, is bounded below), it is natural to expect that G in fact contains a *large* highly connected subgraph. A result of Bohman et al. [1] implies that for every graph G of order n with $\delta(G) \geq 4\sqrt{kn}$, the vertex set $V(G)$ admits a partition such that every part induces a k-connected subgraph of order at least $\sqrt{kn}/2$. In a similar direction, by a recent result of Borozan et al. [4], we know that every graph G of order n with $\delta(G) \geq \sqrt{c(k-1)n}$ contains a k-connected subgraph of order at least $\sqrt{(k-1)n/c}$, where $c = 2123/180$. What if we are interested in finding a larger k-connected subgraph, say of order cn? Along these lines, Bollobás and Gyárfás [3] conjectured that for any graph G of order $n \geq 4k-3$, G or its complement $\overline{G}$ contains a k-connected subgraph H of order at least

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$n-2(k-1)$. Since either G or $\overline{G}$ is a dense graph, we might expect to find a very large highly connected subgraph in one of them. This conjecture was settled affirmatively for $n \geq 13k-15$ by Liu, Morris and Prince [9], and then for $n > 6.5(k-1)$ by Fujita and Magnant [8].

Suppose next that $\delta(G) \geq cn$. Can we find a k-connected subgraph of G on at least cn vertices? It turns out that the answer is "yes" for sufficiently large $n \geq n_0(c,k)$, and in fact a simple argument gives even more — there exists such a subgraph on at least n/m vertices, where $m = \lfloor 1/c \rfloor$. For instance, if $c > 1/2$, then G itself is k-connected. To see the assertion, suppose that G itself is not k-connected. Then G can be split into two "large" pieces with a separating set of size at most $k-1$. Both pieces must have order at least $cn-(k-1)+1$, so as not to violate the minimum degree condition. If one of the pieces does not induce a k-connected subgraph, then we split the subgraph, to obtain another two large pieces and a separating set of size at most $k-1$. Now, each of the three large pieces has order at least $cn-2(k-1)+1$. We repeat this procedure, as long as one of the existing large pieces does not induce a k-connected subgraph. After t steps, we have $t+1$ large pieces, each with order at least $cn-t(k-1)+1$. If $m \leq t = O_n(1)$, then there is a large piece with at most $\frac{n}{m+1} \ll cn$ vertices, which is a contradiction. Thus, the procedure must terminate after $t \leq m-1$ steps, giving us $t+1 \leq m$ large pieces, say $X_1, \ldots, X_t$. Finally, let C be the set of accumulated separating vertices, so that $|C| \leq t(k-1)$. We can "redistribute" the vertices of C to the X_i. The minimum degree condition on G implies that every vertex of C must have at least k neighbours in one of the X_i. Therefore, we can write $C = C_1 \dot{\cup} \cdots \dot{\cup} C_t$ such that, every vertex of C_i has at least k neighbours in X_i, for every $1 \leq i \leq t$. So we are done, since some set $X_i \cup C_i$ induces a k-connected subgraph on at least $n/m \geq cn$ vertices. Furthermore, the value of n/m for the order of the largest k-connected subgraph is sharp, if $k \geq 4$. To see this, we can let G be the union of m cliques, each with $\lceil n/m \rceil + 1$ vertices, and ordered linearly in such a way that any two successive cliques intersect in at most three vertices, with non-consecutive cliques being disjoint.

Here, we instead focus on another graph parameter which forces G to be dense, but which does not immediately yield a trivial bound for our problem. Such a parameter is the independence number $\alpha(G)$. If a graph G has independence number α, then its complement $\overline{G}$ has clique number α, so that, by Turán's theorem, $\overline{G}$ has average degree at most around $(1-1/\alpha)n$, and so G has average degree at least around n/α. It is natural to conjecture that this average degree condition automatically implies that G has a k-connected subgraph on at least n/α vertices. However, this conjecture is false. Indeed, for the cases $\alpha = 2$ and $\alpha = 3$, our graphs in Constructions 5 and 7 (see Section 3) have average degrees $(19/32)n$ and $(307/729)n$, and no k-connected subgraphs of orders $n/2$ and $n/3$ respectively.

Structures in graphs with fixed independence number are widely studied. In particular, the problem of finding a large subgraph with certain properties in a graph with fixed independence number has received much attention. For example, a famous theorem due to Chvátal and Erdős [5] from 1972 states that any graph G on at least three vertices, whose independence number $\alpha(G)$ is at most its connectivity $\kappa(G)$, contains a Hamiltonian cycle. Motivated by this, Fouquet and Jolivet [7] conjectured in 1976 that if, instead, G is a k-connected graph of order n with $\alpha(G) = \alpha \geq k$, then G has a cycle with length at least $\frac{k(n+\alpha-k)}{\alpha}$. Recently, this long standing conjecture was settled affirmatively by O et al. [11].

In this paper, we consider the following question. Fix $k \geq 1$, and let G be a graph on n vertices with independence number α. Can we always find a large k-connected subgraph of G? A little thought shows that, if $n \leq \alpha k$, then there might be no such subgraph, and if $n \geq \alpha k+1$, then we are only guaranteed a k-connected subgraph of order $\lceil n/\alpha \rceil$, since in both cases G might consist of the disjoint union of α cliques, each with either $\lceil n/\alpha \rceil$ or $\lfloor n/\alpha \rfloor$ vertices. Such a graph G has the fewest edges among all graphs of independence number α, so it seems that it should be extremal for our problem as well.

In fact, for large n, this construction (which we will call the *disjoint clique construction*, or just DCC) is indeed extremal. Specifically, we prove in Theorem 2 that any graph G of order $n \geq \alpha^2 k+1$ and independence number α must have a k-connected subgraph of order at least $\lceil n/\alpha \rceil$. However, for smaller values of n, this no longer applies. For instance, when $\alpha = 2$ and $k \geq 3$, there is a graph of order $n = 4k-5$ and independence number 2 with no k-connected subgraph of order at least $\lceil n/2 \rceil$