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# Existence of regular unimodular triangulations of dilated empty simplices

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ABSTRACT

Given integers $k$ and $m$ with $k \geq 2$ and $m \geq 2$, let $P$ be an empty simplex of dimension $(2k-1)$ whose $\delta$-polynomial is of the form $1+(m-1)t^k$. In the present paper, the necessary and sufficient condition for the $k$th dilation $kP$ of $P$ to have a regular unimodular triangulation will be presented.



## 1. Introduction

### 1.1. Integral convex polytopes and $\delta$-polynomials

An *integral* convex polytope is a convex polytope whose vertices are integer points. For an integral convex polytope $P \subset \mathbb{R}^d$ of dimension $d$, we consider the generating function $\sum_{n\geq 0}|nP \cap \mathbb{Z}^d|t^n$, where $nP = \{n\alpha \mid \alpha \in P\}$. Then it is well-known that this becomes a rational function of the form

$$\sum_{n\geq 0}|nP \cap \mathbb{Z}^d|t^n = \frac{\delta_P(t)}{(1-t)^{d+1}},$$

where $\delta_P(t)$ is a polynomial in $t$ of degree at most $d$ with non-negative integer coefficients. The polynomial $\delta_P(t)$ is called the *$\delta$-polynomial*, also known as the *(Ehrhart) $h^*$-polynomial* of $P$. For more details on $\delta$-polynomials of integral convex polytopes, please refer to [2] or [7].

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### 1.2. Empty simplices

An integral simplex $P \subset \mathbb{R}^d$ is called *empty* if $P$ contains no integer point except for its vertices. Note that $P$ is an empty simplex if and only if the linear term of $\delta_P(t)$ vanishes. Empty simplices are of particular interest in the area of not only combinatorics of integral convex polytopes but also toric geometry. Especially, the characterization problem of empty simplices is one of the most important topics. Originally, the empty simplices of dimension 3 were completely characterized by G.K. White [14]. Note that the $\delta$-polynomial of every empty simplex of dimension 3 is of the form $1+(m-1)t^2$ for some positive integer $m$. Recently, this characterization of empty simplices has been generalized by Batyrev–Hofscheier [1]. More precisely, the following theorem has been proved.

**Theorem 1.1** (*cf.* [1, Theorem 2.5]). *Given an integer $k \geq 2$, let $d = 2k-1$. Then $P \subset \mathbb{R}^d$ is an empty simplex of dimension $d$ whose $\delta$-polynomial is of the form $1+(m-1)t^k$ for some $m \geq 2$ if and only if there are integers $a_1, \ldots, a_{k-1}$ with $1 \leq a_i \leq m/2$ and $(a_i, m) = 1$ for each $1 \leq i \leq k-1$ such that $P$ is unimodularly equivalent to the convex hull of*

$$\left\{ \mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \sum_{i=1}^{k-1} a_i \mathbf{e}_i + \sum_{j=k}^{d-1} (m - a_{d-j})\mathbf{e}_j + m\mathbf{e}_d \right\}. \tag{1}$$

Here $(a, b)$ denotes the greatest common divisor of two positive integers $a$ and $b$, $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ are the unit coordinate vectors of $\mathbb{R}^d$ and $\mathbf{0} \in \mathbb{R}^d$ is the origin.

Given integers $a_1, \ldots, a_{k-1}, m$ with $1 \leq a_i \leq m/2$ and $(a_i, m) = 1$ for each $i$, let $P(a_1, \ldots, a_{k-1}, m)$ denote the convex hull of (1).

### 1.3. The integer decomposition property and unimodular triangulations

We say that an integral convex polytope $P \subset \mathbb{R}^d$ has the *integer decomposition property* (IDP, for short) if for each integer $n \geq 1$ and for each $\gamma \in nP \cap \mathbb{Z}^d$, there exist $\gamma^{(1)}, \ldots, \gamma^{(n)}$ belonging to $P \cap \mathbb{Z}^d$ such that $\gamma = \gamma^{(1)} + \cdots + \gamma^{(n)}$.

Under the assumption that the affine lattice generated by $P \cap \mathbb{Z}^d$ is equal to the whole lattice $\mathbb{Z}^d$, the following implications for integral convex polytopes hold:

$$\begin{aligned} &P \text{ has a regular unimodular triangulation} \Rightarrow \text{ a unimodular triangulation} \\ &\qquad \Rightarrow \text{ a unimodular covering } \Rightarrow \text{ IDP.} \end{aligned}$$

(Please refer the reader to [13] for the notions of (regular) unimodular triangulation or unimodular covering.) Note that for each implication, there exists an example of an integral convex polytope not satisfying the converse (see [10], [6] and [3]).

### 1.4. Motivation and results

For any integral convex polytope $P$ of dimension $d$, we know by [4, Theorem 1.3.3] that $nP$ always has IDP for every $n \geq d-1$. Moreover, we also know by [4, Theorem 1.3.1] that there exists a constant $n_0$ such that $nP$ has a unimodular covering for every $n \geq n_0$. However, it is still open whether there really exists a constant $n_0$ such that $nP$ has a (regular) unimodular triangulation for every $n \geq n_0$, while it is only known that there exists a constant $c$ such that $cP$ has a unimodular triangulation [9, Theorem 4.1 (p. 161)].

On the other hand, it is proved in [8] and [12] that for any 3-dimensional integral convex polytope $P$, $nP$ has a unimodular triangulation for $n = 4$ [8] and every $n \geq 6$ ([12, Theorem 1.4]). For the proofs of those results, the discussions about the existence of a (regular) unimodular triangulation of the *dilated* empty simplices of dimension 3 are crucial, where for an empty simplex $P$, a dilated empty simplex means a simplex $nP$ for some positive integer $n$. Hence, for the further investigation of higher-dimensional cases, the existence of a unimodular triangulation of dilated empty simplices might be important. Since $P(a_1, \ldots, a_{k-1}, m)$ can be understood as a generalization of empty simplices of dimension 3, it is quite reasonable to study the existence of a unimodular triangulation of the dilated