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Polynomial expansion and sublinear separators

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ABSTRACT

Let $\mathcal{C}$ be a class of graphs that is closed under taking subgraphs. We prove that if for some fixed $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then any depth-r minor (i.e. minor obtained by contracting disjoint subgraphs of radius at most r) of a graph in $\mathcal{C}$ has average degree $O\big((r \operatorname{polylog} r)^{1/\delta}\big)$. This confirms a conjecture of Dvořák and Norin.

1. Introduction

For an integer $r \geq 0$, a *depth-r minor* of a graph G is a subgraph of a graph that can be obtained from G by contracting pairwise vertex-disjoint subgraphs of radius at most r. Let $d(G)$ denote the average degree of a graph $G = (V, E)$, i.e. $d(G) = 2|E|/|V|$. For some function f, we say that a class $\mathcal{C}$ of graphs has *expansion bounded by f* if for any graph $G \in \mathcal{C}$ and any integer r, any depth-r minor of G has average degree at most $f(r)$. We say that a class has *bounded expansion* if it has expansion bounded by some function f, and *polynomial expansion* if f can be taken to be a polynomial.

Classes of bounded expansion play a central role in the study of sparse graphs [7]. From an algorithmic point of view, a very useful property of these classes is that when their expansion is not too large (say subexponential), graphs in the class have sublinear separators. A *separator* in a graph $G = (V, E)$ is a pair of subsets (A, B) of vertices of G such that $A \cup B = V$ and no edge of G has one endpoint in $A \setminus B$ and the other in $B \setminus A$. The separator (A, B) is said to be *balanced* if both $|A \setminus B|$ and $|B \setminus A|$ contain at most $\frac{2}{3}|V|$ vertices. The *order* of the separator (A, B) is $|A \cap B|$.

A class $\mathcal{C}$ of graphs is *monotone* if for any graph $G \in \mathcal{C}$, any subgraph of G is in $\mathcal{C}$. Dvořák and Norin [5] observed that the following can be deduced from a result of Plotkin, Rao, and Smith [8].

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Theorem 1 (*[5]*). *Let $\mathcal{C}$ be a monotone class of graphs with expansion bounded by $r \mapsto c(r+1)^{1/4\delta-1}$, for some constant $c > 0$ and $0 < \delta \leq 1$. Then there is a constant C such that every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$.*

Dvořák and Norin [5] also proved the following partial converse.

Theorem 2 (*[5]*). *Let $\mathcal{C}$ be a monotone class of graphs such that for some fixed constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by some function $f(r) = O(r^{5/\delta^2})$.*

They conjectured that the exponent $5/\delta^2$ of the polynomial expansion in Theorem 2 could be improved to match (asymptotically) that of Theorem 1.

Conjecture 3 (*[5]*). *There exists a real $c > 0$ such that the following holds. Let $\mathcal{C}$ be a monotone class of graphs such that for some fixed constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by some function $f(r) = O(r^{c/\delta})$.*

In this short note, we prove this conjecture.

Theorem 4. *For any $C > 0$ and $0 < \delta \leq 1$, if a monotone class $\mathcal{C}$ has the property that every n-vertex graph in $\mathcal{C}$ has a balanced separator of order at most $Cn^{1-\delta}$, then $\mathcal{C}$ has expansion bounded by the function $f : r \mapsto c_1 \cdot (r+1)^{1/\delta}(\frac{1}{\delta}\log(r+3))^{c_2/\delta}$, for some constants c_1 and c_2 depending only on $\mathcal{C}$.*

In particular Conjecture 3 holds for any real number $c > 1$. The proof of Theorem 4 is given in the next section, and we conclude with some open problems in Section 3.

2. Proof of Theorem 4

We need the following results. The first is a classical connection between balanced separators and tree-width (see [5]).

Lemma 5. *Any graph G has a balanced separator of order at most* $\mathrm{tw}(G) + 1$.

Dvořák and Norin [4] proved that the following partial converse holds.

Theorem 6 (*[4]*). *If every subgraph of G has a balanced separator of order at most k, then G has tree-width at most $105k$.*

Note that in our proof of Theorem 4 we could also use the weaker (and easier) result of [1] that under the same hypothesis, G has tree-width at most $1+k\log|V(G)|$, but the computation is somewhat less cumbersome if we use Theorem 6 instead.

For a set S of vertices in a graph G, we let $N(S)$ denote the set of vertices not in S with at least one neighbor in S. We will use the following result of Shapira and Sudakov [9].

Theorem 7 (*[9]*). *Any graph G contains a subgraph H of average degree $d(H) \geq \frac{255}{256}d(G)$ such that for any set S of at most $n/2$ vertices of H (where $n = |V(H)|$), $|N(S)| \geq \frac{1}{2^8 \log n(\log\log n)^2}|S|$.*

In fact, we will only need a much weaker version, where the vertex-expansion is of order $\Omega\left(\frac{1}{\mathrm{polylog}\, n}\right)$ instead of $\Omega\left(\frac{1}{\log n(\log\log n)^2}\right)$.

Finally, we need a result of Chekuri and Chuzhoy [3] on bounded-degree subgraphs of large tree-width in a graph of large tree-width.

Theorem 8 (*[3]*). *There are constants α, β such that for any integer $k \geq 2$, any graph G of tree-width at least k contains a subgraph H of tree-width at least $\alpha k/(\log k)^\beta$ and maximum degree 3.*