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Bijections for Weyl Chamber walks ending on an axis, using arc diagrams and Schnyder woods

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ABSTRACT

In the study of lattice walks there are several examples of enumerative equivalences which amount to a trade-off between domain and endpoint constraints. We present a family of such bijections for simple walks in Weyl chambers which use arc diagrams in a natural way. One consequence is a set of new bijections for standard Young tableaux of bounded height. A modification of the argument in two dimensions yields a bijection between Baxter permutations and walks ending on an axis, answering a recent question of Burrill et al. (2016). Some of our arguments (and related results) are proved using Schnyder woods. Our strategy for simple walks extends to any dimension and yields a new bijective connection between standard Young tableaux of height at most $2k$ and certain walks with prescribed endpoints in the k-dimensional Weyl chamber of type D.

1. Introduction

In the context of directed 2D lattice paths with unit steps, there is a classic bijection between *meanders* and *bridges* of equal length. This maps lattice walks with steps $(1, 1)$ and $(1, -1)$ starting at the origin, staying above the x-axis (meanders) to those ending at height zero (bridges)—see Fig. 1. This example illustrates a common trade-off in lattice walks between domain constraints and endpoint constraints [15,6]. Note that the natural bijection shown in Fig. 1 proceeds via an intermediate class

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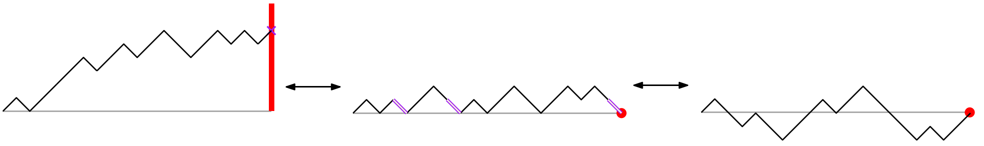

Fig. 1. An example of the classical bijection between meanders and bridges. From a Dyck walk D with some marked steps $d_1, \ldots, d_k$ reaching the x-axis, one gets (bijectively) a meander by turning every marked step into an up-step. To get (bijectively) a bridge from D, for $1 \leq i \leq k$ we let u_i be the up-step matched with d_i, and we switch every step between u_i and d_i (included).

of walks (Dyck walks) where both the stronger domain and endpoint restrictions are imposed, and the elements of this class carry additional "decorations" (here, marked down-steps reaching the x-axis).

This paper introduces a similar but new strategy which can be successfully applied to several models of Weyl chamber walks. In particular, for two classical step sets (simple walks and hesitating walks), we have found explicit bijections that exchange a domain constraint with an endpoint constraint. In the two-dimensional case, these bijections match walks in the quadrant $\{x \geq 0,\ y \geq 0\}$ ending at the origin (*excursions*[1]), and walks in the octant $\{x \geq y \geq 0\}$ and ending on the x-axis (*axis-walks*). For both step sets, these bijections pass through decorated excursions restricted to the octant. Deciding exactly how to mark the steps in the decorated intermediary is less obvious than the Dyck walk example. We do this by using *open arc diagrams* (corresponding to partial matchings and set partitions) that are associated to the walks via the robust bijection of Chen et al. [11]—or more precisely, via its extension to open arc diagrams due to Burrill et al. [8]. In their full generality, these bijections map open arc diagrams with no $(k+1)$-crossing[2] to walks in the k-dimensional Weyl chamber of type C $\{(x_1, \ldots, x_k) : x_1 \geq \cdots \geq x_k \geq 0\}$ that end on the x_1-axis, where the number of open arcs gives the abscissa of the endpoint. It is at the level of arc diagrams that the marking of the object is easiest to describe: we map walks that end on the x-axis to open arc diagrams, mark the location of the open arcs, remove them, and then apply the inverse bijection to get marked excursions. The schematic outline of our core idea is illustrated in Fig. 2. The advantage of our approach is that it very easily generalizes to walks in arbitrary dimension.

Once these decorated excursions are obtained, it remains to process the marks. This processing is handled differently for simple walks and for hesitating walks (where a further step of transfer of decorations is needed), but in both cases the marks are used to produce an unmarked walk in a larger domain.

Part of the bijection for simple walks has the nice feature that it extends to higher dimension, unveiling a new bijective connection with standard Young tableaux of even-bounded height (which are known to be related to Weyl chamber axis-walks [22,8,26]).

1.1. Bijection for 2D simple walks

A lattice model is said to be *simple* if the step set consists of all of the elementary vectors and their negatives. In two dimensions, the steps correspond to the compass directions, that we denote N, E, S, W. Our first main result is the following Theorem, which is proved in Section 3.

Theorem 1. *There exists an explicit bijection (preserving the length) between simple axis-walks of even length staying in the first octant, and simple excursions staying in the first quadrant.*

As announced in the introduction, our strategy uses open arc diagrams to turn the simple axis-walk of length $2n$ into a decorated excursion. This is then transformed to a simple walk of length $2n$ in the tilted quadrant $\{(x, y) : x \geq 0,\ |y| \leq x\}$ starting and ending at $(1/2, 1/2)$, and finally mapped to a pair of Dyck paths of respective lengths $2n+2$ and $2n$. These are known [12,2] to be in bijection with simple excursions of length $2n$ in the quadrant. This chain of bijections is illustrated by Fig. 3.

1 More generally, we use the term excursion to indicate the set of walks with a prescribed start and end point. When they are not specified, the prescribed start and end is assumed to be the origin.

2 A k-crossing is a set of k mutually crossing arcs.