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A new lower bound for van der Waerden numbers

Thomas Blankenship, Jay Cummings, Vladislav Taranchuk

Department of Mathematics and Statistics, Sacramento State University, United States

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ABSTRACT

In this paper we prove a new recurrence relation on the van der Waerden numbers, $w(r, k)$. In particular, if p is a prime and $p \leq k$ then $w(r, k) > p \cdot \left(w\left(r - \left\lceil \frac{r}{p} \right\rceil, k\right) - 1\right)$. This recurrence gives the lower bound $w(r, p+1) > p^{r-1}2^p$ when $r \leq p$, which generalizes Berlekamp's theorem on 2-colorings, and gives the best known bound for a large interval of r. The recurrence can also be used to construct explicit valid colorings, and it improves known lower bounds on small van der Waerden numbers.

1. Introduction and history

In 1927, van der Waerden proved that for any positive integers r and k there exists an $N = w(r, k)$ such that every r-coloring of $\{1, 2, 3, \ldots, N\}$ contains a monochromatic arithmetic progression of length k. As a central function in Ramsey theory and a notoriously difficult one to understand, the growth rate of $w(r, k)$ has received much attention.

Van der Waerden's initial proof gives a monstrous upper bound. In the slowest-growing case, when $r = 2$, still the bound is $w(2, k) \leq A(n)$, where $A(n)$ is the Ackermann function. The best known general upper bound is due to Gowers [8], who proved

$$w(r, k) \leq 2^{2^{r^{2^{2^{k+9}}}}}.$$

In [10] Graham and Solymosi improved this in the case when $k = 3$, which in a series of follow-up papers by Bourgain [3], Sanders [17] and Bloom [2] further improved the upper bound to

$$w(r, 3) \leq 2^{cr(\ln r)^4}$$

E-mail addresses: thomasblankensh@csus.edu (T. Blankenship), jay.cummings@csus.edu (J. Cummings), vtaranchuk@csus.edu (V. Taranchuk).

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where $c > 0$ is an absolute constant. Graham currently offers 1000 USD for an answer as to whether or not $w(2, k) < 2^{k^2}$.

In 1953, Erdős and Rado [6] proved the lower bound

$$\sqrt{2(k-1)r^{k-1}} \leq w(r, k)$$

using a simple counting argument. In 1960, Moser [15] used a constructive approach to improve this bound in the case that r is large relative to k. In particular, he showed that

$$(k-1)r^{C \ln(r)} < w(r, k)$$

for some absolute constant C. Two years later Schmidt [18] used a nonconstructive approach to prove a bound that is asymptotically better in k. He showed that there is some absolute constant c for which

$$r^{k-c\sqrt{k \ln(k)}} \leq w(r, k).$$

In 1968, Berlekamp used an algebraic approach to construct what is still the best known lower bound for the case when $k = p + 1$, where p is a prime, and $r = 2$. He showed that

$$p2^p < w(2, p + 1).$$

In this paper we use a construction to generalize this result to the following.

Theorem 1.1. *If p is any prime with $2 \leq r \leq p \leq k$, then*

$$p^{r-1}2^p < w(r, p + 1).$$

This generalizes Berlekamp's theorem [1].

In 1973, Erdős and Lovász [5] used the Lovász Local Lemma on hypergraphs to show that

$$\frac{r^{k-1}}{4k}\left(1 - \frac{1}{k}\right) \leq w(r, k).$$

In this paper we will use a recurrence to generalize Berlekamp's result to arbitrary number of colors. Our work will also improve bounds on small van der Waerden numbers. Finally, our bound is recursively-constructive, in that an explicit coloring when $r = 2$ can be used to create explicit colorings for larger r.

The current best known general lower bound of this type is due to Kozik and Shabanov [12], who in 2016 proved

$$c \cdot r^{k-1} \leq w(r, k)$$

for some absolute constant $c > 0$.

For large $r \gg k$, the best result, by O'Bryant, can be obtained by using the Hypergraph Symmetry Theorem and the Behrend-type results about sets of integers without long progressions (see [16]):

$$w(r, k) > e^{f(k)(\ln r)^{\lceil \log_2 k \rceil}}$$

where $f(k)$ is a function of k. The above bound can be found in [4] and is best known for large $r \gg k$.

There are now constructive approaches to the Lovász Local Lemma (see [7]), which can be used to produce explicit constructions with high probability. Therefore, in a sense, the above two bounds can also be considered constructive.

In the following section we establish a recursive lower bound for $w(r, k)$, which is used to deduce our main result. In Section 4 we use this recurrence relation to improve known numerical lower bounds for some small values of r and k.

2. Proof of the main theorem

Definition 2.1. Let R_r represent the set of colors $\{1, 2, \ldots, r\}$. For each $i \in R_r$, define $S_i(r, k)$ to be the p-tuple

$$S_i(r, k) = (i, i + 1, i + 2, \ldots, r, 1, 2, 3, \ldots, r, 1, 2, \ldots),$$

where p is the largest prime such that $p \leq k$.