ARTICLE IN PRESS

European Journal of Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

# 1-page and 2-page drawings with bounded number of crossings per edge☆

Carla Binucci [a], Emilio Di Giacomo [a], Md. Iqbal Hossain [b], Giuseppe Liotta [a]

[a] Dipartimento di Ingegneria, Università degli Studi di Perugia, Italy
[b] Bangladesh University of Engineering and Technology, Bangladesh

ARTICLE INFO

*Article history:*
Available online xxxx

ABSTRACT

A drawing of a graph such that the vertices are drawn as points along a line and each edge is a circular arc in one of the two half-planes defined by this line is called a 2-page drawing. If all edges are in the same half-plane, the drawing is called a 1-page drawing. We want to compute 1-page and 2-page drawings of planar graphs such that the number of crossings per edge does not depend on the number of vertices. We show that for any constant $k$, there exist planar graphs that require more than $k$ crossings per edge in both 1-page and 2-page drawings. We then prove that if the vertex degree is bounded by $\Delta$, every planar 3-tree has a 2-page drawing with a number of crossings per edge that only depends on $\Delta$. Finally, we show a similar result for 1-page drawings of partial 2-trees.



## 1. Introduction

A *k-page book embedding* (also known as *stack layout*) of a planar graph $G$ is a crossing-free drawing of $G$ where the vertices are represented as points along a line called *spine* and each edge is a circular arc in one of $k$ half-planes bounded by the spine; each such half-plane is called a *page*. The minimum number of pages to compute a book embedding of a planar graph $G$ is the *book thickness* of $G$. It is known that not all planar graphs have book thickness two; in fact, Bernhart and Kainen proved

☆ This is an extended version of a conference paper Binucci et al. (2016)[6]. Research supported in part by the MIUR project AMANDA "Algorithmics for MAssive and Networked DAta", prot. 2012C4E3KT_001.

*E-mail addresses:* carla.binucci@unipg.it (C. Binucci), emilio.digiacomo@unipg.it (E. Di Giacomo), mdiqbalhossain@cse.buet.ac.bd (M.I. Hossain), giuseppe.liotta@unipg.it (G. Liotta).

http://dx.doi.org/10.1016/j.ejc.2017.07.009
0195-6698/

that a planar graph has page number two if and only if it is sub-Hamiltonian [5]. On the other hand, Yannakakis showed that every planar graph has book thickness at most four [24] and it is an outstanding open problem establishing whether four pages are sometimes necessary [12].

As a consequence, if we want to compute a drawing of a (not necessarily sub-hamiltonian) planar graph such that all vertices are points on a line and each edge is a circular arc inside one of at most two half-planes, edge crossings are unavoidable. If the edges are restricted to lie in exactly one of the two half-planes, we talk about 1-page drawings; otherwise, we talk about 2-page drawings. We remark that the main difference between a 2-page drawing and a 2-page book embedding is that the first type of drawing admits edge crossings while the second does not. Since testing a graph for sub-Hamiltonicity is NP-complete [8], the result by Bernhart and Kainen implies that minimizing the number of edge crossings in a 2-page drawing is also NP-complete. Two recent papers by Bannister et al. [4] and by Bannister and Eppstein [3] show that minimizing the number of edge crossings in a 2-page drawing is fixed parameter tractable with respect to various graph parameters, such as cyclomatic number and treewidth. Crossing minimization heuristics and genetic algorithms for 2-page drawings are provided in [16] and in [14,15], respectively. Recently, Ábrego et al. [1] studied the 2-page crossing number of complete graphs (i.e., the minimum number of crossings determined by a 2-page drawing of a complete graph), and proved that the Harary–Hill conjecture[1] for these types of drawings holds.

In this paper we study the problem of computing 1-page and 2-page drawings of planar graphs such that the number of crossings *per edge* is bounded by a function that does not depend on the size of the graph. We prove the following theorems about planar graphs with bounded treewidth and bounded vertex degree.

**Theorem 1.** *Let $G$ be a planar 3-tree with maximum degree $\Delta$ and $n$ vertices. There exists an $O(n)$-time algorithm that computes a 2-page drawing of $G$ with at most $2\Delta$ crossings per edge. Also, for every integer constant $k$, there exist infinitely many planar 3-trees that do not admit a 2-page drawing with at most $k$ crossings per edge.*

**Theorem 2.** *Let $G$ be a partial 2-tree with maximum degree $\Delta$ and $n$ vertices. There exists an $O(n)$-time algorithm that computes a 1-page drawing of $G$ with at most $\Delta^2$ crossings per edge. Also, for every integer constant $k$, there exist infinitely many partial 2-trees that do not admit a 1-page drawing with at most $k$ crossings per edge.*

*Research context.* The contribution of this paper can be related with a fertile research area of graph drawing devoted to computing drawings where some edge crossing configurations are forbidden. In particular, a graph is said to be $k$-planar if it has a drawing where each edge is crossed at most $k$ times. Theorems 1 and 2 compute $k$-planar 1-page and 2-page drawings where $k$ is a function of $\Delta$. Recent results about $k$-planar graphs and drawings include [2,10,11].

The results are also related with the study of the book thickness of $k$-trees. Bernhart and Kainen [5] prove that 1-trees, i.e. standard trees, have book thickness one, while Rengarajan and Veni Madhavan [20] proved that every 2-tree has book thickness two. Ganley and Heath [13] showed that every $k$-tree has pagenumber at most $k+1$ and conjectured that this bound can be reduced to $k$. Dujmović and Wood [12] disproved this conjecture showing that there exist a $k$-tree that has book thickness $k+1$ for every $k \geq 3$. Other examples of $k$-trees that have book thickness $k+1$ and that satisfy additional properties are presented in Vandenbussche et al. [21] and in Dujmović and Wood [23].

We finally recall that 1-page and 2-page drawings are among the oldest and more common graph drawing conventions and they have received different names during the years. For example, they were studied in the 60s under the name of *network permutations* (see, e.g., [19]); they were called *linear embeddings* in the 90s (see, e.g., [17]); they were introduced in the InfoVis community less than fifteen years ago with the name of *arc diagrams* (see, e.g., [22]). A preliminary version of this paper appears in [6].

*Paper Organization.* The remainder of this paper is organized as follows. Preliminary definitions are given in Section 2. Theorem 1 is proved in Section 3, while Theorem 2 is proved in Section 4. Open problems are discussed in Section 5.

[1] The Harary–Hill conjecture states that the crossing number of the complete graph $K_n$ is $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.