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3-coloring triangle-free planar graphs with a precolored 9-cycle

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ABSTRACT

Given a triangle-free planar graph G and a 9-cycle C in G, we characterize situations where a 3-coloring of C does not extend to a proper 3-coloring of G. This extends previous results when C is a cycle of length at most 8.

1. Introduction

Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We will also use $|G|$ for the size of $E(G)$. A *proper k-coloring* of a graph G is a function $\varphi : V(G) \to \{1, 2, \dots, k\}$ such that $\varphi(u) \neq \varphi(v)$ for each edge $uv \in E(G)$. A graph G is *k-colorable* if there exists a proper k-coloring of G, and the minimum k where G is k-colorable is the *chromatic number* of G.

Garey and Johnson [18] proved that deciding if a graph is k-colorable is NP-complete even when $k = 3$. Moreover, deciding if a graph is 3-colorable is still NP-complete when restricted to planar graphs [12]. Therefore, even though planar graphs are 4-colorable by the celebrated Four Color Theorem [5,6,22], finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research. A landmark result in this area is Grötzsch's Theorem [20], which is the following:

Theorem 1 (*[20]*)**.** *Every triangle-free planar graph is* 3*-colorable.*

We direct the readers to a nice survey by Borodin [8] for more results and conjectures regarding 3-colorings of planar graphs.

A graph G is *k-critical* if it is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. Critical graphs are important since they are (in a certain sense) the minimal obstacles in

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reducing the chromatic number of a graph. Numerous coloring algorithms are based on detecting critical subgraphs. Despite its importance, there is no known characterization of k-critical graphs when $k \geq 4$. On the other hand, there has been some success regarding 4-critical planar graphs. Extending Theorem 1, the Grünbaum–Aksenov Theorem [1,7,21] states that a planar graph with at most three triangles is 3-colorable, and we know that there are infinitely many 4-critical planar graphs with four triangles. Borodin, Dvořák, Kostochka, Lidický, and Yancey [9] were able to characterize all 4-critical planar graphs with four triangles.

Given a graph G and a proper subgraph C of G, we say G is *C-critical for k-coloring* if for every proper subgraph H of G where $C \subseteq H$, there exists a proper k-coloring of C that extends to a proper k-coloring of H, but does not extend to a proper k-coloring of G. Roughly speaking, a C-critical graph for k-coloring is a minimal obstacle when trying to extend a proper k-coloring of C to a proper k-coloring of the entire graph. Note that $(k+1)$-critical graphs are exactly the C-critical graphs for k-coloring with C being the empty graph.

In the proof of Theorem 1, Grötzsch actually proved that any proper coloring of a 4-cycle or a 5-cycle extends to a proper 3-coloring of a triangle-free planar graph. This implies that there are no triangle-free planar graphs that are C-critical for 3-coloring when C is a face of length 4 or 5. This sparked the interest of characterizing triangle-free planar graphs that are C-critical for 3-coloring when C is a face of longer length. Since we deal with 3-coloring triangle-free planar graphs in this paper, from now on, we will write "C-critical" instead of "C-critical for 3-coloring" for the sake of simplicity.

The investigation was first done on planar graphs with girth 5. Walls [25] and Thomassen [23] independently characterized C-critical planar graphs with girth 5 when C is a face of length at most 11. The case when C is a 12-face was initiated in [23], but a complete characterization was given by Dvořák and Kawarabayashi in [13]. Moreover, a recursive approach to identify all C-critical planar graphs with girth 5 when C is a face of any given length is given in [13]. Dvořák and Lidický [17] implemented the algorithm from [13] and used a computer to generate all C-critical graphs with girth 5 when C is a face of length at most 16. The generated graphs were used to reveal some structure of 4-critical graphs on surfaces without short contractible cycles. It would be computationally feasible to generate graphs with girth 5 even when C has length greater than 16.

The situation for planar graphs with girth 4, which are triangle-free planar graphs, is more complicated since the list of C-critical graphs is not finite when C has size at least 6. We already mentioned that there are no C-critical triangle-free planar graphs when C is a face of length 4 or 5. An alternative proof of the case when C is a 5-face was given by Aksenov [1]. Gimbel and Thomassen [19] not only showed that there exists a C-critical triangle-free planar graph when C is a 6-face, but also characterized all of them. A *k^--cycle, k^+-cycle* is a cycle of length at most k, at least k, respectively. A cycle C in a graph G is *separating* if $G - C$ has more connected components than G.

Theorem 2 (*Gimbel and Thomassen [19]*)**.** *Let G be a connected triangle-free plane graph with outer face bounded by a 6^--cycle $C = c_1c_2\cdots$. The graph G is C-critical if and only if C is a 6-cycle, all internal faces of G have length exactly four and G contains no separating 4-cycles. Furthermore, if φ is a 3-coloring of C that does not extend to a 3-coloring of G, then $\varphi(c_1) = \varphi(c_4)$, $\varphi(c_2) = \varphi(c_5)$, and $\varphi(c_3) = \varphi(c_6)$.*

Aksenov, Borodin, and Glebov [3] independently proved the case when C is a 6-face using the discharging method, and also characterized all C-critical triangle-free planar graphs when C is a 7-face in [4]. The case where C is a 7-face was used in [9].

Theorem 3 (*Aksenov, Borodin, and Glebov [4]*)**.** *Let G be a connected triangle-free plane graph with outer face bounded by a 7-cycle $C = c_1\cdots c_7$. The graph G is C-critical and ψ is a 3-coloring of C that does not extend to a 3-coloring of G if and only if G contains no separating 5^--cycles and one of the following propositions is satisfied up to relabeling of vertices (see Fig. 1 for an illustration).*

(a) *The graph G consists of C and the edge c_1c_5, and $\psi(c_1) = \psi(c_5)$.*
(b) *The graph G contains a vertex v adjacent to c_1 and c_4, the cycle $c_1c_2c_3c_4v$ bounds a 5-face and every face drawn inside the 6-cycle $vc_4c_5c_6c_7c_1$ has length four; furthermore, $\psi(c_4) = \psi(c_7)$ and $\psi(c_5) = \psi(c_1)$.*