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Contagious sets in dense graphs☆

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ABSTRACT

We study the activation process in undirected graphs known as bootstrap percolation: a vertex is active either if it belongs to a set of initially activated vertices or if at some point it had at least r active neighbors, for a threshold r that is identical for all vertices. A contagious set is a vertex set whose activation results with the entire graph being active. Let $m(G, r)$ be the size of a smallest contagious set in a graph G on n vertices.

We examine density conditions that ensure $m(G, r) = r$ for all r, and first show a necessary and sufficient condition on the minimum degree. Moreover, we study the speed with which the activation spreads and provide tight upper bounds on the number of rounds it takes until all nodes are activated in such graphs.

We also investigate what average degree asserts the existence of small contagious sets. For $n \geq k \geq r$, we denote by $M(n, k, r)$ the maximum number of edges in an n-vertex graph G satisfying $m(G, r) > k$. We determine the precise value of $M(n, k, 2)$ and $M(n, k, k)$, assuming that n is sufficiently large compared to k.

1. Introduction

In this article we study the *r-neighbor bootstrap percolation* process. Here we are given an undirected graph $G = (V, E)$ and an integer $r \geq 1$. Every vertex is either *active* or *inactive*. We say a set A of vertices is active if all vertices in A are active. The vertices that are active initially are called

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seeds, and the set of seeds is denoted by A_0. If vertices become active thereafter we also refer to them as *infected*. A contagious process evolves in discrete rounds. The set of active vertices in round $i > 0$ is

$$A_i = A_{i-1} \cup \{v : |N(v) \cap A_{i-1}| \geq r\},$$

where $N(v)$ is the set of neighbors of v. That is, a vertex becomes active irrevocably in a given round if it has at least r active neighbors. We refer to r as the *threshold* of the graph. Let $\langle A_0 \rangle$ be the set of nodes that will eventually become infected if we activate A_0.

Definition 1. Given $G = (V, E)$, a set $A_0 \subseteq V$ is called *contagious* (also known as "percolating") if $\langle A_0 \rangle = V$. In words, activating A_0 results in the infection of the entire vertex set. The size of the smallest contagious set is denoted by $m(G, r)$. For a contagious set A_0, the number of rounds until total infection is the smallest integer t with $A_t = V$. We call t the *infection time* for A_0.

The term bootstrap percolation is used sometimes to model the case where the seeds are chosen independently at random. In this work we use this term also with respect to the deterministic selection of a contagious set. Bootstrap percolation was first studied by the statistical physicists Chalupa, Leath, and Reich [11]. Since then, this model has found applications in many fields. For example, this model is related to "word of mouth" effects occurring in viral marketing, where the information is only revealed to a small group of persons initially, who subsequently share it with their friends resulting in a cascade that may spread to the entire network. Similarly, we can think of cascading effects in finance, where an institute might default if a certain number of business partners fail (cp. [15,2,16,12] and the references therein). Furthermore, various questions related to bootstrap percolation have been examined for a large variety of graphs including work on hypercubes by Balogh and Bollobás [4] and on grids by Balogh, Bollobás, Duminil-Copin, and Morris [5], and by Balogh and Pete [6]. Several models of random graphs were studied by Janson, Łuczak, Turova, and Vallier [14], by Amini and Fountoulakis [3], and by Balogh and Pittel [7]. Coja-Oghlan, Feige, Krivelevich, and Reichman examined contagious sets in expanders [13].

A natural question is to determine for a given integer k, what combinatorial properties of graphs ensure that there is a contagious set of size k. Such a characterization seems difficult even for $k = 2$ (and $r = 2$). Indeed, the family of all graphs with a contagious set of size two include, for example, cliques, bipartite cliques (with both sides larger than one), and binomial random graphs with edge probability $p \geq n^{-1/2+\epsilon}$ [14].

Ackerman, Ben-Zwi, and Wolfovitz [1] and Reichman [19] examined the connection between $m(G, r)$ and the degree sequence of G. Here we continue this line of investigation and study two basic (and interrelated) graph parameters: the minimum degree and edge cardinality. More concretely, our goal is to determine what conditions on these parameters imply that $m(G, r) \leq k$ where k is small compared to the number of vertices in G, and $r \leq k$. We study the cases $r = k$ and $r = 2$.

How large does the minimum degree have to be in order to guarantee a contagious set of size $k = r \geq 2$? Clearly, such a contagious set has minimum cardinality if it exists. We prove that $\lceil \frac{k-1}{k} \cdot n \rceil$ suffices, where n is the number of vertices. In particular, if $k = 2$ then the required minimum degree is $\lceil \frac{n}{2} \rceil$. A graph with this property is called Dirac graph. We also show that this condition on the minimum degree is the best possible. For $k = 2$ this is easy to see: if we lower the minimum degree to $\lceil \frac{n}{2} - 1 \rceil$, then G may be disconnected implying that $m(G, 2) > 2$ (provided that G has at least three vertices). In Section 5 we demonstrate that a contagious set of size 2 also exists in a generalization of Dirac graphs known as Ore graphs (Dirac graphs and Ore graphs are known to have a Hamiltonian cycle).

While the minimum size of contagious sets has been studied thoroughly, much less is known on the number of rounds that the activation process takes to infect the whole graph (e.g., see the work of Bollobás, Holmgren, Smith, and Uzzell [9], and the articles of Bollobás, Smith, and Uzzell [10] and of Przykucki [18]). We define the *maximum infection time* (also referred to as maximum percolation time) of a graph G as the largest infection time for any of its contagious sets (cp. Definition 1), i.e. the largest number of rounds that the activation process requires until the whole graph is infected. The maximum infection time of a class of graphs is defined analogously.

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