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A characterization of tightly triangulated 3-manifolds

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ABSTRACT

For a field $\mathbb{F}$, the notion of $\mathbb{F}$-tightness of simplicial complexes was introduced by Kühnel. Kühnel and Lutz conjectured that $\mathbb{F}$-tight triangulations of a closed manifold are the most economic of all possible triangulations of the manifold.

The boundary of a triangle is the only $\mathbb{F}$-tight triangulation of a closed 1-manifold. A triangulation of a closed 2-manifold is $\mathbb{F}$-tight if and only if it is $\mathbb{F}$-orientable and neighbourly. In this paper we prove that a triangulation of a closed 3-manifold is $\mathbb{F}$-tight if and only if it is $\mathbb{F}$-orientable, neighbourly and stacked. In consequence, the Kühnel–Lutz conjecture is valid in dimension ≤ 3.

1. Introduction

All simplicial complexes considered in this paper are finite and abstract. The vertex set of a simplicial complex X will be denoted by $V(X)$. For $A \subseteq V(X)$, the induced subcomplex $X[A]$ of X on vertex set A is defined by $X[A] := \{\alpha \in X : \alpha \subseteq A\}$. For $x \in V(X)$, the subcomplexes $\mathrm{ast}_X(x) = \{\alpha \in X : x \notin \alpha\} = X[V(X) \setminus \{x\}]$ and $\mathrm{lk}_X(x) = \{\alpha \in X : x \notin \alpha, \alpha \sqcup \{x\} \in X\}$ are called the *antistar* and the *link* of x in X, respectively. A simplicial complex X is said to be a *triangulated (closed) manifold* if it triangulates a (closed) manifold, i.e., if the geometric carrier $|X|$ of X is a (closed) topological manifold. A triangulated closed and connected d-manifold X is said to be $\mathbb{F}$-*orientable* if $H_d(X; \mathbb{F}) \neq 0$. If two triangulated d-manifolds X and Y intersect in a single common d-face α then

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$X\#Y := (X \cup Y) \setminus \{\alpha\}$ is called the *connected sum of X and Y along* α. It triangulates the topological connected sum $|X|\#|Y|$ or $|X|\#\overline{|Y|}$. Notice that in dimension ≤ 3 these two spaces are homeomorphic.

For our purpose, a *graph* may be defined as a simplicial complex of dimension ≤ 1. For $n \geq 3$, the *n-cycle* C_n is the unique n-vertex connected graph in which each vertex lies in exactly two edges. For $n \geq 1$, the *complete graph* K_n is the n-vertex graph in which each pair of vertices forms an edge. For $m, n \geq 1$, the *complete bipartite graph* $K_{m,n}$ is the graph with $m + n$ vertices and mn edges in which each of the first m vertices forms an edge with each of the last n vertices. Two graphs are said to be *homeomorphic* if their geometric carriers are homeomorphic. A graph is said to be *planar* if it is a subcomplex of a triangulation of the 2-sphere S^2. In this paper, we shall have an occasion to use the easy half of Kuratowski's famous characterization of planar graphs [5]: A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

If $\mathbb{F}$ is a field and X is a simplicial complex then, following Kühnel [9], we say that X is $\mathbb{F}$-tight if (a) X is connected, and (b) the $\mathbb{F}$-linear map $H_*(Y;\mathbb{F}) \to H_*(X;\mathbb{F})$, induced by the inclusion map $Y \hookrightarrow X$, is injective for every induced subcomplex Y of X.

If X is a simplicial complex of dimension d, then its *face vector* $(f_0, \ldots, f_d)$ is defined by $f_i = f_i(X) := \#\{\alpha \in X : \dim(\alpha) = i\}$, $0 \leq i \leq d$. A simplicial complex X is said to be *neighbourly* if each pair of its vertices forms an edge, i.e., if $f_1(X) = \binom{f_0(X)}{2}$.

A simplicial complex X is said to be *strongly minimal* if, for every triangulation Y of the geometric carrier $|X|$ of X, we have $f_i(X) \leq f_i(Y)$ for all i, $0 \leq i \leq \dim(X)$. Our interest in the notion of $\mathbb{F}$-tightness mainly stems from the following famous conjecture.

Conjecture 1.1 (*Kühnel–Lutz [10]*)**.** *For any field $\mathbb{F}$, every $\mathbb{F}$-tight triangulated closed manifold is strongly minimal.*

Following Walkup [16] and McMullen–Walkup [12], a triangulated ball B is said to be *stacked* if all the faces of B of codimension 2 are contained in the boundary ∂B of B. A triangulated sphere S is said to be *stacked* if there is a stacked ball B such that $S = \partial B$. This notion was extended to triangulated manifolds by Murai and Nevo [14]. Thus, a triangulated manifold X with boundary ∂X is said to be *stacked* if all its faces of codimension 2 are contained in ∂X. A triangulated closed manifold M is said to be *stacked* if there is a stacked triangulated manifold X such that $M = \partial X$. A triangulated manifold is said to be *locally stacked* if all its vertex links are stacked spheres or stacked balls.

The main result of this paper is the following characterization of $\mathbb{F}$-tight triangulated closed 3-manifolds, for all fields $\mathbb{F}$.

Theorem 1.2. *A triangulated closed 3-manifold M is $\mathbb{F}$-tight if and only if M is $\mathbb{F}$-orientable, neighbourly and stacked.*

The special case of Theorem 1.2, where $\mathrm{char}(\mathbb{F}) \neq 2$, was proved in our previous paper [4]. In [4, Conjecture 1.12] we conjectured the validity of Theorem 1.2 in general. In fact, it is already known from [3] that an $\mathbb{F}$-tight closed triangulated 3-manifold M is $\mathbb{F}$-orientable and neighbourly. Here, we show that it is also stacked.

As a consequence of Theorem 1.2, we show that the Kühnel–Lutz conjecture (Conjecture 1.1) is valid up to dimension 3. Thus,

Corollary 1.3. *If M is an $\mathbb{F}$-tight triangulated closed manifold of dimension ≤ 3, then M is strongly minimal.*

As a second consequence of Theorem 1.2, we show:

Corollary 1.4. *The only closed topological 3-manifolds which may possibly have $\mathbb{F}$-tight triangulations are S^3, $(S^2 \mathbin{\underline{\times}} S^1)^{\#k}$ and $(S^2 \times S^1)^{\#k}$, where k is a positive integer such that $80k + 1$ is a perfect square.*

The boundary of the 4-simplex is a tight triangulation of S^3. Moreover, in [7], $\mathbb{Z}_2$-tight triangulations of $(S^2 \mathbin{\underline{\times}} S^1)^{\#k}$ are constructed for $k = 1, 30, 99, 208, 357$ and 546. The smallest values of k for which no tight triangulation of $(S^2 \mathbin{\underline{\times}} S^1)^{\#k}$ is known are $k = 12, 19, 21$. In contrast, no $\mathbb{F}$-tight triangulations of $(S^2 \times S^1)^{\#k}$ can be found in the literature.