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Minimum size of feedback vertex sets of planar graphs of girth at least five

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ABSTRACT

A feedback vertex set of a graph is a subset of vertices intersecting all cycles. We provide tight upper bounds on the size of a minimum feedback vertex set in planar graphs of girth at least five. We prove that if G is a connected planar graph of girth at least five on n vertices and m edges, then G has a feedback vertex set of size at most $\frac{2m-n+2}{7}$. By Euler's formula, this implies that G has a feedback vertex set of size at most $\frac{m}{5}$ and $\frac{n-2}{3}$. These results not only improve a result of Dross, Montassier and Pinlou and confirm the girth-5 case of one of their conjectures, but also make the best known progress towards a conjecture of Kowalik, Lužar and Škrekovski and solve the subcubic case of their conjecture. An important step of our proof is providing an upper bound on the size of minimum feedback vertex sets of subcubic graphs with girth at least five with no induced subdivision of members of a finite family of non-planar graphs.

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1. Introduction

In this paper, graphs are simple. A *feedback vertex set* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ is a forest. We define $\phi(G)$ to be the minimum size of a feedback vertex set of a graph G. Feedback vertex sets have been extensively studied. For example, given a graph G and an integer k, deciding if $\phi(G) \leq k$ is one of Karp's original NP-complete problems [6].

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A direction in the study of feedback vertex sets is to find an upper bound on the minimum size of a feedback vertex set (for example, see [2,9,10]). An old conjecture due to Albertson and Berman [1] states that every planar graph on n vertices has a feedback vertex set of size at most $\frac{n}{2}$. This conjecture, if true, implies that every planar graph on n vertices has an independent set of size at least $\frac{n}{4}$. This bound for the independent set is an immediate corollary of the Four Color Theorem, but it is the only proof in the literature. Albertson and Berman's conjecture remains open. The best known result is that every planar graph G on n vertices has a feedback vertex set with size at most $\frac{3n}{5}$, due to Borodin's acyclic 5-coloring theorem [3].

The *girth* of a graph is the length of its shortest cycle. (If the graph has no cycle, then its girth is infinity.) Recently, the minimum size of feedback vertex sets of planar graphs with girth at least five has attracted attention. Kowalik et al. [8] proposed the following conjecture, which is tight as the dodecahedron attains the bound.

Conjecture 1.1 *([8]). If G is a planar graph of girth at least five on n vertices, then $\phi(G) \leq \frac{3n}{10}$.*

Dross et al. [4] proved that if G is a planar graph of girth at least five with m edges, then $\phi(G) \leq \frac{5m}{23}$. Note that every planar graph on n vertices with girth at least five has at most $\frac{5}{3}(n-2)$ edges by Euler's formula (when $n \geq 4$). So every planar graph with girth at least five on n vertices has a feedback vertex set of size at most $\frac{25n-50}{69}$ (when $n \geq 4$).

In a companion paper, Dross et al. [5] made the following conjecture.

Conjecture 1.2 *([5]). If G is a planar graph of girth at least g with m edges, then $\phi(G) \leq \frac{m}{g}$.*

In this paper, we prove the following.

Theorem 1.3. *If G is a connected planar graph of girth at least five on n vertices and m edges, then $\phi(G) \leq \frac{2m-n+2}{7}$.*

A graph is *subcubic* if every vertex has degree at most three. It is easy to see that every subcubic graph on n vertices has at most $\frac{3n}{2}$ edges. Together with the fact that every non-tree planar graph on n vertices with girth at least five has at most $\frac{5}{3}(n-2)$ edges, the following are immediate corollaries of Theorem 1.3.

Corollary 1.4. *Let G be a graph on n vertices and m edges with $n \geq 2$.*

1. *If G is a planar graph with girth at least five, then $\phi(G) \leq \frac{m}{5}$, and $\phi(G) \leq \frac{n-2}{3}$.*
2. *If G is a planar subcubic graph with girth at least five, then $\phi(G) \leq \frac{3n}{10}$.*
3. *If G is a connected planar subcubic graph with girth at least five, then $\phi(G) \leq \frac{2n+2}{7}$.*

The first statement of Corollary 1.4 improves the result of Dross et al. about Conjecture 1.1 mentioned earlier and solves Conjecture 1.2 for the case $g = 5$. The second or the third statement of Corollary 1.4 solves Conjecture 1.1 for the case when G is subcubic. Note that the dodecahedron attains the bound in Theorem 1.3.

Our strategy for proving Theorem 1.3 is first proving the case when G is a subcubic graph, and then boosting it to the general case. In fact, instead of proving the subcubic case of Theorem 1.3, we will prove a stronger result: the planarity will be replaced by the property of the lack of induced subdivision of graphs in a finite family of non-planar graphs. We say that a graph G contains an *induced subdivision* of another graph H if G contains an induced subgraph that can be obtained from H by repeatedly subdividing edges. For every even number n with $n \geq 6$, let M_n be the cubic graph obtained from the n-cycle by adding edges connecting each opposite pair of vertices. Notice that M_n is not planar but any of its proper induced subgraphs is. So unlike the minor relation or the subdivision relation, there is no Kuratowski-type theorem for planarity with respect to the induced subdivision relation, even if we restrict the problem to subcubic graphs.

Furthermore, in order to make the induction go through, we relax the girth condition to allow the existence of short cycles, but we do not allow two disjoint short cycles. The following is our result about subcubic graphs. The family $\mathcal{F}$ and the function r mentioned in the description will be explicitly described in this paper.

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