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# All binomial identities are orderable

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ABSTRACT

The main result of this paper is to show that all binomial identities are orderable. This is a natural statement in the combinatorial theory of finite sets, which can also be applied in distributed computing to derive new strong bounds on the round complexity of the weak symmetry breaking task.

Furthermore, we introduce the notion of a fundamental binomial identity and find an infinite family of values, other than the prime powers, for which no fundamental binomial identity can exist.



## 1. Preliminaries

For any natural number $n$, we set $[n] := \{1, \dots, n\}$.

**Definition 1.1.** A *binomial identity* is any equality of the form

$$\binom{n}{a_1} + \cdots + \binom{n}{a_k} = \binom{n}{b_1} + \cdots + \binom{n}{b_m}, \tag{1.1}$$

where $n$ is a natural number and $0 \le a_1 < \cdots < a_k \le n$, $0 \le b_1 < \cdots < b_m \le n$, $a_i \ne b_j$, $\forall i, j$.

Note, that each binomial identity is associated to a fixed value of $n$. One handy way to phrase the identity (1.1) is to consider index sets $A := \{a_1, \dots, a_k\}$ and $B := \{b_1, \dots, b_m\}$, and then to say that the number of ways to choose from the set with $n$ elements a subset, whose cardinality lies in $A$, is the same as the number of ways to choose from this set a subset, whose cardinality lies in $B$. For future reference, we also introduce notation for the families of subsets $\mathcal{A} := \{S \subseteq [n] \mid |S| \in A\}$ and $\mathcal{B} := \{T \subseteq [n] \mid |T| \in B\}$.

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The binomial identity then simply says that $|\mathcal{A}| = |\mathcal{B}|$, i.e., there exists a bijection between $\mathcal{A}$ and $\mathcal{B}$.

**Definition 1.2.** We say that the binomial identity (1.1) is *orderable* if there exists a bijection $\Phi : \mathcal{A} \to \mathcal{B}$, such that for each $S \in \mathcal{A}$ we either have $S \subseteq \Phi(S)$ or $S \supseteq \Phi(S)$.[1]

One may view $\mathcal{A}$ and $\mathcal{B}$ as subsets of the Boolean algebra $\mathcal{C}^n$, consisting of entire levels indexed by $A$ and $B$. The binomial identity is then orderable if and only if there is a perfect matching between elements of $\mathcal{A}$ and elements of $\mathcal{B}$, such that we are allowed only to match comparable elements. Our main result says that this can always be done.

**Theorem 1.3.** *All binomial identities are orderable.*

We note a special binomial identity $\binom{n}{k} = \binom{n}{n-k}$, for which Theorem 1.3 is well-known and many explicit bijections have been constructed, e.g., using Catalan factorization of walks. Before we can give our proof of the main theorem, we need to make some constructions and to recall a few facts.

We start with some graph terminology. Given a graph $G$, we let $V(G)$ denote its set of vertices, and we let $E(G)$ denote its set of edges. For a vertex $v \in V(G)$, we set $N(v) := \{w \in V(G) \mid (v, w) \in E(G)\}$, the set of all vertices *adjacent* to $v$. We extend this notation to sets of vertices $S \subseteq V(G)$ by setting $N(S) := \bigcup_{v \in S} N(v)$, so $N(S)$ is the set of all vertices of $G$ adjacent to *some* vertex of $S$.

A graph $G$ is called *bipartite* if its set of vertices can be split as a disjoint union $V(G) = U \cup W$, such that every edge of $G$ has one vertex in $U$ and one vertex in $W$. We shall say that $G = (U, W)$ is a *bipartite split*; note that it may not be unique if the graph is not connected. Note that if $S \subseteq U$, then $N(S) \subseteq W$ and vice versa.

**Definition 1.4.** Assume we are given two disjoint collections of subsets $\mathcal{X}, \mathcal{Y} \subseteq 2^{[n]}$. We let $\Gamma_{\mathcal{X},\mathcal{Y}}$ denote the bipartite graph defined as follows:

- the vertices are the sets in these collections: $V(\Gamma_{\mathcal{X},\mathcal{Y}}) = \mathcal{X} \cup \mathcal{Y}$;
- the sets $S \in \mathcal{X}$ and $T \in \mathcal{Y}$ are connected by an edge if and only if $S \subset T$ or $T \subset S$.

As said above, a binomial identity is orderable if and only if the bipartite graph $\Gamma_{\mathcal{A},\mathcal{B}}$ has a perfect matching. The *matching theory* is a rich theory, and the following theorem provides a standard criterion for the existence of a perfect matching, see e.g., [3,9].

**Theorem 1.5** (*Hall's Marriage Theorem*)**.** *Assume $G = (A, B)$ is a bipartite graph, such that $|A| = |B|$. The graph $G$ has a perfect matching if and only if for every set $Z \subseteq A$ we have*

$$|N(Z)| \geq |Z|. \tag{1.2}$$

In addition to the graph terminology, we need some combinatorial notions related to Boolean algebra. For all $0 \leq k \leq n$, we let $\mathcal{C}^n_k := \{S \subseteq [n] \mid |S| = k\}$ denote the $k$th level in the Boolean algebra $\mathcal{C}^n$.

**Definition 1.6.** Assume we are given $\mathcal{S} \subseteq \mathcal{C}^n_a$ and $0 \leq b \leq n$. We set

$$\mathrm{Sh}_b(\mathcal{S}) := \begin{cases} \{T \in \mathcal{C}^n_b \mid T \subseteq S, \text{ for some } S \in \mathcal{S}\}, & \text{if } b \leq a; \\ \{T \in \mathcal{C}^n_b \mid T \supseteq S, \text{ for some } S \in \mathcal{S}\}, & \text{if } b \geq a. \end{cases}$$

Furthermore, for any set $B \subseteq \{0, \ldots, n\}$, we set $\mathrm{Sh}_B(\mathcal{S}) := \bigcup_{b \in B} \mathrm{Sh}_b(\mathcal{S})$. We call these sets *b-shadow* and *B-shadow* of $\mathcal{S}$, respectively.

We adopted here the standard terminology from Sperner theory, see [1, Chapter 2], though we do not distinguish between shadows and shades. Clearly, in terms of the bipartite graph $\Gamma_{\mathcal{X},\mathcal{Y}}$ above, the shadow operation coincides with the adjacency operation $N(-)$.

1 The author is not aware of any previous appearance of this definition in the literature.