Journal of Combinatorial Theory, Series A 161 (2019) 1–28

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Promotion on generalized oscillating tableaux and web rotation

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A R T I C L E   I N F O

Article history:
Received 18 September 2017
Available online xxxx

Keywords:
Webs
Promotion
Tableaux

A B S T R A C T

We introduce the notion of a generalized oscillating tableau and define a promotion operation on such tableaux that generalizes the classical promotion operation on standard Young tableaux. As our main application, we show that this promotion corresponds to rotation of the irreducible A_2-webs of G. Kuperberg.

1. Introduction

Recall that a *partition* is a finite, nonincreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_t)$ and that any partition can be identified with the corresponding *Young diagram*—a left-justified array of boxes with λ_i boxes in the ith row from the top. An *oscillating tableau* of length k is a sequence of $k+1$ partitions $(\lambda^0 = \varnothing, \ldots, \lambda^k)$, where $\lambda^0 = \varnothing$ and λ^i is obtained from λ^{i-1} by either adding or deleting one box. In this paper, we generalize these notions.

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https://doi.org/10.1016/j.jcta.2018.07.005
0097-3165/

We define a *generalized partition with* n *parts* $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ to be a nonincreasing list of n (not necessarily positive) integers. We introduce the notion of a *generalized oscillating tableau of length* k *with* n *parts*: a sequence of $k+1$ generalized partitions $(\varnothing, \lambda^1, \ldots, \lambda^k)$ such that each λ^i has n parts, $\lambda^0 = \varnothing = (0, \ldots, 0)$, and λ^{i+1} can be obtained from λ^i by either adding or subtracting 1 from one of $\lambda_1^i, \ldots, \lambda_n^i$. We visualize generalized partitions using a generalization of Young diagrams, where we allow negative row sizes and indicate negative rows by coloring the corresponding boxes red. In this visualization, the number of parts is the number of rows in the diagram. In particular, we leave space for an empty row for each part of size 0. We may then associate a set-valued tableau T to each generalized oscillating tableau, where the set of boxes of T is the union of boxes in $\lambda^1, \ldots, \lambda^k$ and we add entry i (resp. i') to the subset of primed and unprimed positive integers in a box if λ^i is obtained from λ^{i+1} by adding (resp. deleting) the corresponding box. For example, the generalized oscillating tableau of length 5 with 2 parts

$$((0,0),(1,0),(1,-1),(2,-1),(2,0),(1,0))$$

corresponds to the set-valued filling below.

$$\begin{array}{ccc} & \boxed{1} & \boxed{35'} \\ \boxed{2'4} & & \end{array}$$

Let $\mathrm{GOT}(k,n)$ denote the set of generalized oscillating tableaux of length k with n parts. We define a promotion operation $p : \mathrm{GOT}(k,n) \to \mathrm{GOT}(k,n)$ that generalizes classical tableau promotion. We define this promotion operation using both growth rules and growth diagrams and using tableau rules. Fig. 1 shows an example of generalized oscillating promotion. A reader familiar with promotion on standard Young tableaux will recognize the similarities. Our promotion is related to the growth diagrams of Akhmejanov [1].

As our main application, we relate generalized oscillating promotion on $\mathrm{GOT}(k,3)$ to rotation of irreducible A_2-webs. An *irreducible* A_2*-web* can be defined as a bipartite graph with fixed coloring embedded in a disk such that each vertex on the boundary of the disk has degree 1, each interior vertex has degree 3, and all internal faces have at least 6 sides. Webs were defined by G. Kuperberg motivated by the study of multilinear invariant theory [7]. In his paper, Kuperberg introduces combinatorial rank 2 spiders, which are a diagrammatic presentation of the space $\mathrm{Inv}(V_1 \otimes \cdots \otimes V_n)$, i.e., the invariant space of a tensor product of irreducible representations V_i of a rank 2 Lie algebra $\mathfrak{g}$. Webs are a basis for the invariant space in this diagrammatic presentation.

Webs have since been studied by M. Khovanov and G. Kuperberg [5]; T.K. Peterson, P. Pylyavskyy, and B. Rhoades [8]; S. Fomin and P. Pylyavskyy [3]; and many others. In particular, Khovanov and Kuperberg describe a bijection between webs and *dominant signature and state strings*: certain vectors of pairs, where each pair $(j_i, s_i) \in \{\bullet, \circ\} \times \{1, 0, \bar{1}\}$.