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Hyperplane equipartitions plus constraints

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ABSTRACT

While equivariant methods have seen many fruitful applications in geometric combinatorics, their inability to answer the now settled Topological Tverberg Conjecture has made apparent the need to move beyond the use of Borsuk–Ulam type theorems alone. This impression holds as well for one of the most famous problems in the field, dating back to 1960, which seeks the minimum dimension $d := \Delta(m; k)$ such that any m mass distributions in $\mathbb{R}^d$ can be simultaneously equipartitioned by k hyperplanes. Precise values of $\Delta(m; k)$ have been obtained in few cases, and the best-known general upper bound $U(m; k)$ typically far exceeds the conjectured-tight lower bound arising from degrees of freedom. Following the "constraint method" of Blagojević, Frick, and Ziegler originally used for Tverberg-type results and recently to the present problem, we show how the imposition of further conditions – on the hyperplane arrangements themselves (e.g., orthogonality, prescribed flat containment) and/or the equipartition of additional masses by successively fewer hyperplanes ("cascades") – yields a variety of optimal results for constrained equipartitions of m mass distributions in dimension $U(m; k)$, including in dimensions *below* $\Delta(m+1; k)$, which are still extractable via equivariance. Among these are families of exact values for full orthogonality as well as cascades which maximize the "fullness" of the equipartition at each stage, including some strengthened equipartitions in dimension $\Delta(m; k)$ itself.

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1. Introduction

1.1. Historical summary

With the recent negative resolution [3] to the Topological Tverberg Conjecture [2], perhaps the most famous remaining open question in topological combinatorics is the hyperplane mass equipartition problem, originating with Grünbaum [12] in 1960 and generalized by Ramos [18] in 1996:

Question 1. *[Grünbaum–Ramos] What is the minimum dimension $d := \Delta(m,k)$ such that any m mass distributions $\mu_1,\dots,\mu_m$ on $\mathbb{R}^d$ can be simultaneously equipartitioned by k hyperplanes?*

By a mass distribution μ on $\mathbb{R}^d$, one means a positive, finite Borel measure such that any hyperplanes has measure zero (e.g., if μ is absolutely continuous with respect to Lebesgue measure). To say that k hyperplanes $H_1,\dots,H_k$ equipartition μ means that

$$\mu(\mathcal{R}_g) = \frac{1}{2^k}\mu(\mathbb{R}^d) \tag{1.1}$$

for all $g = (g_1,\dots,g_k) \in \mathbb{Z}_2^{\oplus k}$, where the $\mathcal{R}_g := H_1^{g_1} \cap \cdots \cap H_k^{g_k}$ are the regions obtained by intersecting the resulting half-spaces $H_i^0 := \{\mathbf{u} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{a}_i\rangle \geq b_i\}$ and $H_i^1 := \{\mathbf{u} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{a}_i\rangle \leq b_i\}$ determined by each hyperplane $H_i = \{\mathbf{u} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{a}_i\rangle = b_i\}$, $(\mathbf{a}_i, b_i) \in S^{d-1} \times \mathbb{R}$. Note that equipartitioning hyperplanes are distinct (and affine independent), lest $\mathcal{R}_g = \emptyset$ for some $g \in \mathbb{Z}_2^{\oplus k}$.

The lower bound

$$k\Delta(m;k) \geq m(2^k - 1) \tag{1.2}$$

was proved by Ramos [18] via a generalization of a moment curve argument of Avis [1] for $m = 1$, and the conjecture $\Delta(m;k) = L(m;k) := \left\lceil \frac{m(2^k-1)}{k} \right\rceil$ posited there has been confirmed for all known values of $\Delta(m;k)$. Owing to the reflective and permutative symmetries on k hyperplanes, there is a natural action of the wreath product $\mathfrak{S}_k^{\pm} := \mathbb{Z}_2 \wr \mathfrak{S}_k$ on each collection of regions, $\mathfrak{S}_k$ being the symmetric group, so that upper bounds on $\Delta(m;k)$ have been obtained via equivariant topology. Using the ubiquitous "Configuration-Space/Test-Map (CS/TM) paradigm" formalized by Živaljević (see, e.g., [24]), any collection of k equipartitioning hyperplanes can be identified with a zero of an associated continuous $\mathfrak{S}_k^{\pm}$-map $f : X \to V$, where X is either the k-fold product or join of spheres and V is a certain $\mathfrak{S}_k^{\pm}$-module (see Section 3 for a review of this construction). In favorable circumstances the vanishing of such maps is guaranteed by Borsuk–Ulam type theorems which rely on the calculation of advanced algebraic invariants such as the ideal-valued index theory of Fadell–Husseini [11] or relative equivariant obstruction