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Partitioning the Boolean lattice into copies of a poset

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Let P be a poset of size 2^k that has a greatest and a least element. We prove that, for sufficiently large n, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P. This resolves a conjecture of Lonc.

1. Introduction

Let $2^{[n]}$ denote the *Boolean lattice* of dimension n, that is, the poset (partially ordered set) whose elements are the subsets of $[n] = \{1, \dots, n\}$, ordered by inclusion.

An important property of the Boolean lattice is that any finite poset P can be embedded into $2^{[n]}$ for sufficiently large n. (Perhaps the simplest way to see this is by embedding P into $2^{[|P|]}$ in the following way: we identify the elements of P with numbers $1, \dots, |P|$ and map each $x \in P$ to the set $\{y \in P : y \leq_P x\}$.) Here by an *embedding* of a poset P into a poset Q we mean an injection $f : P \to Q$ such that $f(x) \leq_Q f(y)$ if and only if $x \leq_P y$. For any embedding $f : P \to Q$, we call the image $f(P)$ a *copy* of P in Q.

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Now, if P is fixed and n is large, then $2^{[n]}$ contains many copies of P. So a natural question arises: can $2^{[n]}$ be partitioned into copies of P? Of course, for such a partition to exist, the size of P must divide the size of $2^{[n]}$, that is, $|P|$ must be a power of 2 (we would like to emphasise that we denote by $|P|$ the number of elements of P and not the number of relations). Moreover, P must have a greatest and a least element. Lonc [8] conjectured that these obvious necessary conditions are in fact sufficient for n large enough.

Conjecture 1 *(Lonc). Let P be a poset of size 2^k with a greatest and a least element. Then, for sufficiently large n, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

The case where P is a chain of size 2^k was originally conjectured by Sands [9]. Griggs [3] proposed a slightly stronger conjecture that, for any positive integer c and for sufficiently large n, it is possible to partition $2^{[n]}$ into chains of length c and at most one other chain. Both conjectures were proved by Lonc [8]. The question of minimising the dimension n in Griggs' conjecture in terms of the length of the chain c has received attention from several authors, including Elzobi and Lonc [1] and Griggs, Yeh and Grinstead [4]. Recently, Tomon [11] proved that the smallest sufficient n is of order $\Theta(c^2)$. Related questions on partitioning $2^{[n]}$ into chains of almost equal lengths have also been examined, by Füredi [2], Hsu, Logan, Shahriari and Towse [6,7] and Tomon [10].

As we mentioned in the previous paragraph, Lonc himself verified Conjecture 1 in the case where P is a chain. Furthermore, it is easy to extend this result to products of chains. In fact, for any two posets P, Q, if $2^{[n]}$ can be partitioned into copies of P and $2^{[m]}$ can be partitioned into copies of Q, then $2^{[n+m]}$ can be partitioned into copies of $P \times Q$. However, apart from some small cases that can be checked by hand, chains and their products were the only two cases for which Lonc's conjecture had been confirmed.

In this paper we resolve the conjecture in full generality.

Theorem 2. *Let P be a poset of size 2^k with a greatest and a least element. Then, for sufficiently large n, the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.*

The plan of the paper is as follows. In Section 2 we give the most important definitions and outline the structure of the proof of Theorem 2. We give the actual proof in Sections 3 and 4: Section 3 contains a general argument, which works in various settings where a partition of a product set into smaller sets is sought, and might be of independent interest; Section 4 contains ideas that are particular to partitioning $2^{[n]}$ into copies of a fixed poset. Finally, in Section 5 we give some open problems.