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Non-spanning lattice 3-polytopes ☆

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ABSTRACT

We completely classify non-spanning 3-polytopes, by which we mean lattice 3-polytopes whose lattice points do not affinely span the lattice. We show that, except for six small polytopes (all having between five and eight lattice points), every non-spanning 3-polytope P has the following simple description: $P \cap \mathbb{Z}^3$ consists of either (1) two lattice segments lying in parallel and consecutive lattice planes or (2) a lattice segment together with three or four extra lattice points placed in a very specific manner.

From this description we conclude that all the empty tetrahedra in a non-spanning 3-polytope P have the same volume and they form a triangulation of P, and we compute the h^*-vectors of all non-spanning 3-polytopes.

We also show that all spanning 3-polytopes contain a unimodular tetrahedron, except for two particular 3-polytopes with five lattice points.

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1. Introduction and statement of results

A lattice d-polytope is a polytope $P \subset \mathbb{R}^d$ with vertices in $\mathbb{Z}^d$ and with $\operatorname{aff}(P) = \mathbb{R}^d$. We call *size* of P its number of lattice points and *width* the minimum length of the image $f(P)$ when f ranges over all affine non-constant functionals $f : \mathbb{R}^d \to \mathbb{R}$ with $f(\mathbb{Z}^d) \subseteq \mathbb{Z}$. That is, the minimum lattice distance between parallel hyperplanes that enclose P.

In our papers [2–4] we have enumerated all lattice 3-polytopes of size 11 or less and of width greater than one. This classification makes sense thanks to the following result [2, Theorem 3]: *for each $n \in \mathbb{N}$ there are only finitely many lattice 3-polytopes of width greater than one and with exactly n lattice points.* Here and in the rest of the paper we consider lattice polytopes modulo *unimodular equivalence* or *lattice isomorphism*. That is, we consider P and Q *isomorphic* (and write $P \cong Q$) if there is an affine automorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ with $f(\mathbb{Z}^d) = \mathbb{Z}^d$ and $f(P) = Q$.

As a by-product of the classification we noticed that most lattice 3-polytopes are "lattice-spanning", according to the following definition:

Definition. Let $P \subset \mathbb{R}^d$ be a lattice d-polytope. We call *sublattice index* of P the index, as a sublattice of $\mathbb{Z}^d$, of the affine lattice generated by $P \cap \mathbb{Z}^d$. P is called *lattice-spanning* if it has index 1. We abbreviate sublattice index and lattice-spanning as *index* and *spanning*.

In this paper we completely classify non-spanning lattice 3-polytopes. Part of our motivation comes from the recent results of Hofscheier et al. [6,7] on h^*-vectors of spanning polytopes (see Theorem 7.1). In particular, in Section 7 we compute the h^*-vectors of all non-spanning 3-polytopes and show that they still satisfy the inequalities proved by Hofscheier et al. for spanning polytopes, with the exception of empty tetrahedra that satisfy them only partially.

In dimensions 1 and 2, every lattice polytope contains a unimodular simplex, i.e., a lattice basis, and is hence lattice-spanning. In dimension 3 it is easy to construct infinitely many lattice polytopes of width 1 and of any index $q \in \mathbb{N}$, generalizing White's empty tetrahedra ([9]). Indeed, for any positive integers p, q, a, b with $\gcd(p, q) = 1$ the lattice tetrahedron

$$T_{p,q}(a, b) := \operatorname{conv}\{(0, 0, 0), (a, 0, 0), (0, 0, 1), (bp, bq, 1)\}$$

has index q, width 1, size $a + b + 2$ and volume abq (see a depiction of it in Fig. 1). Here and in the rest of the paper we consider the volume of lattice polytopes normalized to the lattice, so that it is always an integer and the normalized volume of a simplex $\operatorname{conv}(v_0, \dots, v_d)$ equals its determinant $\left|\det \begin{pmatrix} v_0 \cdots v_d \\ 1 \cdots 1 \end{pmatrix}\right|$.

Lemma 1.1 *(Corollary 3.3). Every non-spanning 3-polytope of width one is isomorphic to some $T_{p,q}(a, b)$.*