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Proof of a conjecture of Kenyon and Wilson on semicontiguous minors

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A B S T R A C T

Kenyon and Wilson showed how to test if a circular planar electrical network with n nodes is well-connected by checking the positivity of $\binom{n}{2}$ central minors of the response matrix. Their test is based on the fact that any contiguous minor of a matrix can be expressed as a Laurent polynomial in the central minors. Moreover, the Laurent polynomial is the generating function of domino tilings of a weighted Aztec diamond. They conjectured that a larger family of minors, semicontiguous minors, can also be written in terms of domino tilings of a region on the square lattice. In this paper, we present a proof of the conjecture.

1. Introduction

The study of the electrical networks comes from classical physics with the work of Ohm and Kirchhoff more than 100 years ago. The *circular planar electrical networks* were first studied systematically by Colin de Verdière [6] and Curtis, Ingerman, Moores,

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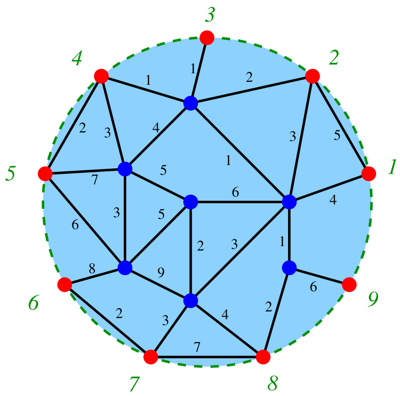

Fig. 1.1. A circular planar electrical network with 9 nodes.

and Morrow [7,8]. Recently, a number of new properties of the circular planar electrical networks have been discovered (see e.g. [1,13,14,23,24,28]).

A *circular planar electrical network* (or simply *network* in this paper) is a finite graph $G = (V, E)$ embedded on a disk with a set of distinguished vertices $N \subseteq V$ on the circle, called *nodes*, and a *conductance function* $wt : E \to \mathbb{R}^+$ (see Fig. 1.1 for an example).

Arrange the indices $1, 2, \dots, n$ of an $n \times n$ matrix $M = (m_{i,j})_{1 \le i,j \le n}$ in counter-clockwise order around the circle. Assume that $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_\ell\}$ are two sets of indices so that $a_1, a_2, \dots, a_k$ and $b_\ell, b_{\ell-1}, \dots, b_1$ are in counter-clockwise order around the circle. We denote by M_A^B the submatrix $\left(m_{a_i,b_j}\right)_{\substack{1 \le i \le k \\ 1 \le j \le \ell}}$ of M. In the case $k = \ell$, we call the pair (A, B) a *circular pair* of M and the determinant $\det M_A^B$ a *circular minor*[2] of M. If A and B are non-interlaced around the circle, we call the latter minor a *non-interlaced circular minor*.

Associated with a network with n nodes is a *response matrix* $\Lambda = (\lambda_{i,j})_{1 \le i,j \le n}$ that measures the response of the network to potential applied at the nodes. In particular, $-\lambda_{i,j}$ is the current that would flow into node j if node i is set to one volt and the remaining nodes are set to zero volts. It has been shown that a matrix M is the response matrix of a network if and only if it is symmetric with row and column sums equal to zero, and each non-interlaced circular minor $\det M_A^B$ is non-negative (see Theorem 4 in [7]).

A network is called *well-connected* if for any two non-interlaced sets of k nodes A and B, there are k pairwise vertex-disjoint paths in G connecting the nodes in A to the nodes in B. A number of equivalent definitions of the well-connected networks were given in [6]. It has been shown by Colin de Verdière that a network is well-connected if and only if all non-interlaced circular minors of the response matrix are positive.

[2] In this paper, we refer *minors* as determinants of submatrices.