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A family of extremal hypergraphs for Ryser's conjecture

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ABSTRACT

Ryser's Conjecture states that for any r-partite r-uniform hypergraph, the vertex cover number is at most $r-1$ times the matching number. This conjecture is only known to be true for $r \leq 3$ in general and for $r \leq 5$ if the hypergraph is intersecting. There has also been considerable effort made for finding hypergraphs that are extremal for Ryser's Conjecture, i.e. r-partite hypergraphs whose cover number is $r-1$ times its matching number. Aside from a few sporadic examples, the set of uniformities r for which Ryser's Conjecture is known to be tight is limited to those integers for which a projective plane of order $r-1$ exists.

We produce a new infinite family of r-uniform hypergraphs extremal to Ryser's Conjecture, which exists whenever a projective plane of order $r-2$ exists. Our construction is flexible enough to produce a large number of non-isomorphic extremal hypergraphs. In particular, we define what we call the *Ryser poset* of extremal intersecting r-partite r-uniform

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hypergraphs and show that the number of maximal and minimal elements is exponential in $\sqrt{r}$.
This provides further evidence for the difficulty of Ryser's Conjecture.

1. Introduction

A *cover* of a hypergraph is a set of vertices meeting every edge of the hypergraph. The vertex *cover number* $\tau(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the number of vertices in the smallest cover of $\mathcal{H}$. A *matching* is a set of disjoint edges, and the *matching number* $\nu(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the maximum size of a matching consisting of edges of $\mathcal{H}$. A hypergraph with $\nu(\mathcal{H}) = 1$ is called *intersecting*.

A hypergraph is r-uniform if every edge has r vertices. Any r-uniform hypergraph $\mathcal{H}$ satisfies the inequality $\tau(\mathcal{H}) \le r\nu(\mathcal{H})$, since the union of the edges of a maximum matching is a cover. This bound is sharp, as shown by the family of all subsets of size r in a ground set of size $kr-1$ which has $\nu = k-1$ and $\tau = (k-1)r$. There is another sharp example for $\nu = 1$: any r-uniform hypergraph consisting of the lines of some projective plane of order $r-1$ (denoted by $\mathcal{P}_r$). To obtain an example for arbitrary ν, one can take the union of disjoint copies of $\mathcal{P}_r$. A hypergraph is *r-partite* if its vertex set V can be partitioned into r sets $V_1, \ldots, V_r$, called the *sides* of the hypergraph, so that every edge contains at most one vertex from each side. A conjecture commonly attributed to Ryser (but which first appeared in a thesis by his student Henderson [11,5]), asserts that the upper bound $\tau(\mathcal{H}) \le r\nu(\mathcal{H})$ can be improved if the hypergraph is r-partite:

Conjecture 1.1. *For any r-partite r-uniform hypergraph we have*

$$\tau(\mathcal{H}) \le (r-1)\nu(\mathcal{H}). \tag{1}$$

When $r = 2$, Ryser's Conjecture is equivalent to König's Theorem. The only other known general case of the conjecture is $r = 3$, which was proved by Aharoni [3]. However, the conjecture is also known to be true for some special cases. In particular, it has been proven by Tuza [15] for r-partite intersecting hypergraphs when $r \le 5$, and by Francetić, Herke, McKay, and Wanless [8] for $r \le 9$, when one makes the further assumption that any two edges of the r-partite hypergraph intersect in exactly one vertex.

Besides trying to prove the conjecture, there has also been considerable effort in understanding which hypergraphs are extremal for Ryser's Conjecture, i.e. finding r-partite hypergraphs $\mathcal{H}$ with $\tau(\mathcal{H}) = (r-1)\nu(\mathcal{H})$. We call such an object an *r-Ryser hypergraph* (or, without specifying its uniformity, a *Ryser hypergraph*). Denoted by $\mathcal{T}_r$, the *truncated projective plane* of uniformity r is obtained from $\mathcal{P}_r$ by the removal of a single vertex v and the lines containing v. The sides $V_1, \ldots, V_r$ of $\mathcal{T}_r$ are the sets of vertices other than