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Restricted Stirling and Lah number matrices and their inverses

John Engbers [a,1], David Galvin [b,2], Cliff Smyth [c,3]

[a] Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee WI, United States of America
[b] Department of Mathematics, University of Notre Dame, Notre Dame IN, United States of America
[c] Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro NC, United States of America

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ABSTRACT

Given $R \subseteq \mathbb{N}$ let ${n \brace k}_R$, ${n \brack k}_R$, and $L(n,k)_R$ count the number of ways of partitioning the set $[n] := \{1, 2, \ldots, n\}$ into k non-empty subsets, cycles and lists, respectively, with each block having cardinality in R. We refer to these as the R-restricted Stirling numbers of the second kind, R-restricted unsigned Stirling numbers of the first kind and the R-restricted Lah numbers, respectively. Note that the classical Stirling numbers of the second kind, unsigned Stirling numbers of the first kind, and Lah numbers are ${n \brace k} = {n \brace k}_{\mathbb{N}}$, ${n \brack k} = {n \brack k}_{\mathbb{N}}$ and $L(n,k) = L(n,k)_{\mathbb{N}}$, respectively.
It is well-known that the infinite matrices $[{n \brace k}]_{n,k \geq 1}$, $[{n \brack k}]_{n,k \geq 1}$ and $[L(n,k)]_{n,k \geq 1}$ have inverses $[(-1)^{n-k}{n \brack k}]_{n,k \geq 1}$, $[(-1)^{n-k}{n \brace k}]_{n,k \geq 1}$ and $[(-1)^{n-k}L(n,k)]_{n,k \geq 1}$ respectively. The inverse matrices $[{n \brace k}_R]^{-1}_{n,k \geq 1}$, $[{n \brack k}_R]^{-1}_{n,k \geq 1}$ and $[L(n,k)_R]^{-1}_{n,k \geq 1}$ exist and have integer entries if and only if $1 \in R$. We express each entry of each of these matrices as the difference between the cardinalities of two explicitly defined families of labeled forests. In particular the entries of

E-mail address: cdsmyth@uncg.edu (C. Smyth).

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$[\left\{ {n \atop k} \right\}_{[r]}]^{-1}_{n,k\geq 1}$ have combinatorial interpretations, affirmatively answering a question of Choi, Long, Ng and Smith from 2006. If we have $1, 2 \in R$ and if for all $n \in R$ with n odd and $n \geq 3$, we have $n \pm 1 \in R$, we additionally show that each entry of $[\left\{ {n \atop k} \right\}_R]^{-1}_{n,k\geq 1}$, $[\left[{n \atop k} \right]_R]^{-1}_{n,k\geq 1}$ and $[L(n,k)_R]^{-1}_{n,k\geq 1}$ is up to an explicit sign the cardinality of a single explicitly defined family of labeled forests. With R as before we also do the same for restriction sets of the form $R(d) = \{d(r-1)+1 : r \in R\}$ for all $d \geq 1$. Our results also provide combinatorial interpretations of the kth Whitney numbers of the first and second kinds of $\Pi^{1,d}_n$, the poset of partitions of $[n]$ that have each part size congruent to 1 mod d.

1. Introduction

For all integers $n, k \geq 1$, let $\left\{ {n \atop k} \right\}$, $\left[{n \atop k} \right]$, and $L(n,k)$ be the classical Stirling numbers of the second kind, unsigned Stirling number of the first kind, and Lah numbers, respectively. These numbers are defined as follows: $\left\{ {n \atop k} \right\}$ is the number of partitions of $[n] := \{1, 2, \ldots, n\}$ into k non-empty subsets, $\left[{n \atop k} \right]$ is the number of partitions of $[n]$ into k non-empty cyclically ordered sets, i.e. cycles, and $L(n,k)$ is the number of partitions of $[n]$ into k non-empty linearly ordered sets, i.e. lists. All of our partitions will be unordered unless we specify otherwise. Let $S_2 := [\left\{ {n \atop k} \right\}]_{n,k\geq 1}$, $S_1 := [\left[{n \atop k} \right]]_{n,k\geq 1}$, and $L := [L(n,k)]_{n,k\geq 1}$ be infinite matrices with rows and columns indexed by the natural numbers $\mathbb{N} := \{1, 2, \ldots\}$. In this notation n is the row index and k is the column index. It is well-known that $S_2^{-1} = [(-1)^{n-k}\left[{n \atop k} \right]]_{n,k\geq 1}$, $S_1^{-1} = [(-1)^{n-k}\left\{ {n \atop k} \right\}]_{n,k\geq 1}$ and $L^{-1} = [(-1)^{n-k}L(n,k)]_{n,k\geq 1}$. In particular, each entry of each inverse matrix has, up to sign, a combinatorial interpretation.

We consider the following generalizations of Stirling and Lah numbers.

Definition 1.1. For $R \subseteq \mathbb{N}$ the *R-restricted Stirling number of the second kind*, $\left\{ {n \atop k} \right\}_R$, is the number of partitions of $[n]$ into k non-empty subsets such that the cardinality of each subset is restricted to lie in R. Analogously, the *R-restricted unsigned Stirling numbers of the first kind* $\left[{n \atop k} \right]_R$ and *R-restricted Lah numbers* $L(n,k)_R$ are the numbers of partitions of $[n]$ into k cycles and lists, respectively, with cardinalities restricted to lie in R.

Note that we recover the classical Stirling numbers of both kinds and the Lah numbers by taking R to be $\mathbb{N}$ (e.g. $\left\{ {n \atop k} \right\}_{\mathbb{N}} = \left\{ {n \atop k} \right\}$ etc.).

Various instances of restricted numbers have appeared in the literature. Comtet [8, page 222] introduced *r-associated Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}_R$ with $R = \{r, r+1, r+2, \ldots\}$, and obtained recurrence relations and generating functions for them. Belbachir and Bousbaa [2] studied *r-associated Lah numbers*, $L(n,k)_R$ also with $R =$