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Tight bounds on the coefficients of partition functions via stability ☆

Ewan Davies [a], Matthew Jenssen [a], Will Perkins [b], Barnaby Roberts [a]

[a] *London School of Economics, United Kingdom of Great Britain and Northern Ireland*
[b] *University of Birmingham, United Kingdom of Great Britain and Northern Ireland*

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A B S T R A C T

Partition functions arise in statistical physics and probability theory as the normalizing constant of Gibbs measures and in combinatorics and graph theory as graph polynomials. For instance the partition functions of the hard-core model and monomer-dimer model are the independence and matching polynomials respectively.

We show how stability results follow naturally from the recently developed occupancy method for maximizing and minimizing physical observables over classes of regular graphs, and then show these stability results can be used to obtain tight extremal bounds on the individual coefficients of the corresponding partition functions.

As applications, we prove new bounds on the number of independent sets and matchings of a given size in regular graphs. For large enough graphs and almost all sizes, the bounds are tight and confirm the Upper Matching Conjecture of Friedland, Krop, and Markström and a conjecture of Kahn on independent sets for a wide range of parameters. Additionally we prove tight bounds on the number of q-colorings of cubic graphs with a given number of mono-

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E-mail addresses: maths@ewandavies.org (E. Davies), matthew.jenssen@maths.ox.ac.uk (M. Jenssen), math@willperkins.org (W. Perkins), b.j.roberts@lse.ac.uk (B. Roberts).

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chromatic edges, and tight bounds on the number of independent sets of a given size in cubic graphs of girth at least 5.

1. Introduction

The matching polynomial (or matching generating function) of a graph G is the function

$$Z_G^{\mathrm{match}}(\lambda) = \sum_{M \in \mathcal{M}(G)} \lambda^{|M|},$$

where the sum is over $\mathcal{M}(G)$, the set of all matchings in the graph G. Analogously, the independence polynomial of a graph G is

$$Z_G^{\mathrm{ind}}(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|},$$

where $\mathcal{I}(G)$ is the set of all independent sets of G. In statistical physics, $Z_G^{\mathrm{match}}(\lambda)$ and $Z_G^{\mathrm{ind}}(\lambda)$ are the partition functions of the monomer-dimer and hard-core models respectively.

The following theorems give a tight upper bound on $Z_G^{\mathrm{match}}(\lambda)$ and $Z_G^{\mathrm{ind}}(\lambda)$ over the family of all d-regular graphs.

Theorem 1 *(Davies, Jenssen, Perkins, Roberts [9]). For any d-regular graph G and any $\lambda > 0$,*

$$\frac{1}{|V(G)|} \log Z_G^{\mathrm{match}}(\lambda) \le \frac{1}{2d} \log Z_{K_{d,d}}^{\mathrm{match}}(\lambda).$$

Theorem 2 *(Kahn [21,22], Galvin–Tetali [17], Zhao [32]). For any d-regular graph G and any $\lambda > 0$,*

$$\frac{1}{|V(G)|} \log Z_G^{\mathrm{ind}}(\lambda) \le \frac{1}{2d} \log Z_{K_{d,d}}^{\mathrm{ind}}(\lambda).$$

In particular, if we set $\lambda = 1$, then the two theorems say that if $2d$ divides n, then the total number of matchings and the number of independent sets in any d-regular graph on n vertices is at most that of the graph $H_{d,n}$ consisting of $n/2d$ copies of $K_{d,d}$. For much more on such extremal problems for regular graphs, see the notes of Galvin [16], survey of Zhao [33], and paper of Csikvári [5].

In this paper we will address two strengthenings of results of the form above and how they are related. The first possible strengthening of Theorems 1 and 2 is that $H_{d,n}$ might